tion of relative motion of the two frames. Equality of distance in each frame measured perpendicular to the direction of relative motion gives us a clean way to compare clocks in the two latticeworks. Let a flash of light bounce back and forth between a mirror mounted on the rocket reference clock and a mirror mounted on the $y = 1$ rocket clock directly above the reference clock. This flash will return to the origin every two meters of rocket light-travel time. We can trace the path of this flash of light in the laboratory frame up to the same y coordinate and back down again. Using the equality of the speed of light in the two frames we can calculate the laboratory time corresponding to the 2-meter round-trip time in the rocket frame. In the next section this study will lead to a demonstration of the invariance of the interval.

5. Invariance of the Interval

Looking for measure of separation AB that has same value in all inertial frames

Distance between two town gates is calculated from the difference between the x coordinates of the two gates and the difference between the y coordinates. How does one find the analogous physical quantity, the spacetime interval between two events? And between what two events shall this interval be evaluated?

Let *event A* be the *emission* of a flash of light. Let *event B* be the *reception* of this flash after its reflection from another object. All that matters in the end is the pair of events. Neither the light nor the object that reflects it is of any direct interest. Nevertheless, an analysis of the track of this pulse through spacetime reveals quickly and simply a quantity (the interval) that is associated with the two events and that has the same value in all inertial reference frames.

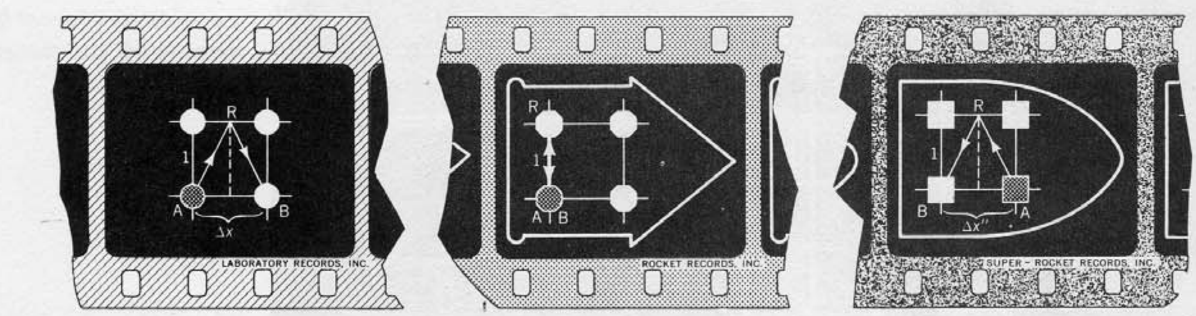
Event A: flash emission
Event B: flash reception

Event A: A spark plug fires. A flash of light flies up to reflector R in Fig. 13. Then the flash wings down. *Event B:* The flash is recorded. Now for the details (Figure 13).

The spark plug fires in the laboratory frame at the zero of time and at the origin of the x, y, z coordinate system (crosshatched). The rocket passes by with such timing that it records the spark as taking place also at *its* origin (likewise crosshatched) and at *its* zero of time. So much for the coordinates of the event of emission:

Details: lab and rocket coordinates of events A and B

$$x_{\text{emission}} = 0, \quad y_{\text{emission}} = 0, \quad t_{\text{emission}} = 0 \quad \text{(laboratory frame)}$$
$$x'_{\text{emission}} = 0, \quad y'_{\text{emission}} = 0, \quad t'_{\text{emission}} = 0 \quad \text{(rocket frame)}$$



A. Light path as observed in laboratory frame

B. Light path as observed in rocket frame

C. Light path as observed in super-rocket frame

Fig. 13. Emission and reflection of the reference flash, and its reception at origin of rocket frame.

The reflector is mounted on the rocket clock 1 meter directly above the origin.

The reception of the flash occurs in the rocket frame at the same place as the emission. The light flash travels a round-trip path of 2 meters. This trip requires 2 meters of light-travel time. The coordinates of the event of reception in the rocket frame are therefore

$$x'_{\text{reception}} = 0, \qquad y'_{\text{reception}} = 0, \qquad t'_{\text{reception}} = 2 \text{ meters}$$

More relevant than an absolute coordinate is the difference in coordinates between the event of reception and the event of emission

$$\begin{aligned} \Delta x' &= x'_{\text{reception}} - x'_{\text{emission}} = 0 \\ \Delta y' &= y'_{\text{reception}} - y'_{\text{emission}} = 0 \\ \Delta t' &= t'_{\text{reception}} - t'_{\text{emission}} = 2 \text{ meters} \end{aligned}$$

In the *laboratory* frame the light flash is received, not at the origin, but at the distance Δx to the right of the origin. High rocket speed implies a large Δx; low rocket speed, a small Δx. (The distance is shown as 1 meter in the figure, but the following analysis is correct for any distance.) In the laboratory frame the flash travels the hypotenuse of two right triangles. Each has a base of $(\Delta x/2)$ and an altitude of 1 meter. The total length of the light path is

$$2[1 + (\Delta x/2)^2]^{1/2}$$

Recall that the speed of light is the same in the laboratory frame as in the rocket frame—the preposterous-but-true character of nature! Therefore the time difference between emission and reception in the laboratory frame is given by the identical formula

$$\Delta t = t_{\text{reception}} - t_{\text{emission}} = 2[1 + (\Delta x/2)^2]^{1/2} \tag{4}$$

in meters of light-travel time.

Time between A and B has different values for lab and rocket observers

Why is this time greater than 2 meters? Because the hypotenuse of a right triangle in Fig. 13,A, is greater than its altitude! There is no escape from the conclusion that *the time between emission and reception is not the same in the two reference frames.*

Both time and space differences between the event of reception and the event of emission are summarized in Table 5.

Table 5. Difference in coordinates between the event of reception and the event of emission.

Laboratory frame	*Rocket frame*
$x_{\text{reception}} - x_{\text{emission}} = \Delta x$	$x'_{\text{reception}} - x'_{\text{emission}} = \Delta x' = 0$
$t_{\text{reception}} - t_{\text{emission}} = \Delta t = 2[1 + (\Delta x/2)^2]^{1/2}$	$t'_{\text{reception}} - t'_{\text{emission}} = \Delta t' = 2$ meters

The time lapse is different in the two reference frames; and so is the space separation—just as the coordinates Δx and Δy of the separation between two town gates are different for the Daytime and Nighttime surveyors! However, for the surveyors there was a combination of coordinates—the square of the *distance* between gates—that was the same for both of them

$$(\text{distance})^2 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + (\Delta y')^2$$

Is there any similar combination of coordinates for two events that will have the same value in the laboratory and rocket frames? Answer: Yes! the square of the *interval*

Interval between A and B has same value for lab and rocket observers

(5) $$(\text{interval})^2 = (\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2 = (2 \text{ meters})^2$$

as one checks directly by substituting in the quantities listed in Table 5.

The rocket frame chosen in which to analyze these two events is a rather special one, in that both emission and reception occur at the same place in it. Figure 13,C, shows the path of the reflected light in a second rocket frame ("super-rocket frame") that is moving even faster relative to the laboratory frame than is the first rocket. In this second rocket frame the difference between the x coordinates for emission and reception (double primes on symbols) $x''_{\text{reception}} - x''_{\text{emission}} = \Delta x''$ is a *negative* quantity because the reception occurs on the negative x axis in this frame. Nevertheless, $(-\Delta x'')^2 = (\Delta x'')^2$ and we can still use the right triangles in Fig. 13,C, to show that the total length of the light path in this second rocket frame is given by the expression $2[1 + (\Delta x''/2)^2]^{1/2}$—which is the same in form as that for the laboratory frame. The speed of light must have the same value in the second rocket frame as in the first rocket frame. Therefore the time between emission and reception is given by

$$t''_{\text{reception}} - t''_{\text{emission}} = \Delta t'' = 2[1 + (\Delta x''/2)^2]^{1/2}$$

Therefore

Interval AB has same value in all rocket frames!

$$(\Delta t'')^2 - (\Delta x'')^2 = (2 \text{ meters})^2$$

also, and in summary

(6) $$(\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2 = (\Delta t'')^2 - (\Delta x'')^2 = (2 \text{ meters})^2$$

Now forget the outgoing light flash, the reflector, and the returning light flash. They were only tools. They helped to identify the quantity that has the same value in different frames of reference. From now on focus on the quantity itself, the interval. Disregard the details of the derivation.

What is the same in two inertial frames; what is nearly the same; and what is different

What has been learned? Two events, A and B, occur at the same point in the rocket frame ($\Delta x' = 0$) but at different times ($\Delta t' = 2$ meters). Viewed in the laboratory frame, those same two events are separated in space by a distance Δx—the faster the rocket happens to be moving, the greater the distance. This result is hardly surprising. One is even entitled to say, "What could be more obvious!" The surprise comes elsewhere. First, *the time* Δt between the two events as recorded in the laboratory frame *does not have the same value as it has in the rocket frame*. Second, the time between A and B as punched out by the two relevant recording laboratory clocks is *greater* than the time between the same two events as recorded by the indentical reference clock of the rocket: $\Delta t \geqslant \Delta t'$. Third, the factor of increase of the time (see Table 5)

$$\Delta t/\Delta t' = [1 + (\Delta x/2)^2]^{1/2}$$

is close to unity (that is, the increase itself is very small) if the distance Δx covered by the rocket between events A and B is small. However, if the rocket moves very fast, Δx is a very great quantity, and the factor of discrepancy between the two times is enormous. Fourth, despite this newly found difference in *time* as recorded in the two reference frames, and despite the long-known difference between the *space* separation of the events in the two reference frames ($\Delta x \neq \Delta x' = 0$), there is nevertheless a quantity that *is* the same in the laboratory frame as the 2 meters of elapsed light-travel time between A and B in the rocket frame. This quantity is the interval,

$$(\text{interval}) = [(\Delta t)^2 - (\Delta x)^2]^{1/2}$$

The rocket speed may be very high. Then Δx will be very large. But then Δt is also very large. Moreover, the magnitude of Δt is perfectly tailored to the size of Δx. In consequence, the special quantity $(\Delta t)^2 - (\Delta x)^2$ has the value (2 meters)^2, no matter how great Δx and Δt individually may be.

One diagram illustrates the four great ideas of special relativity

All of these relationships can be seen at a glance in Fig. 13,A. The hypotenuse of the first right triangle is $\Delta t/2$. Its base is $\Delta x/2$. To say that $(\Delta t)^2 - (\Delta x)^2$ has a standard value, and thus to state that $(\Delta t/2)^2 - (\Delta x/2)^2$ has a standard value, is to say that the altitude of this right triangle has a fixed magnitude (1 meter in the diagram) no matter how fast the rocket is going. What then was the keystone of the argument establishing the fact that $(\Delta t)^2 - (\Delta x)^2$ has the value (2 meters)^2, no matter how fast the rocket is moving? The keystone was the principle of relativity, according to which there is no difference in the laws of physics between one inertial reference frame and another. This principle was put to use here in two very different ways. First, it was used to reason that distances at right angles to the direction of relative motion are recorded as of equal magnitude in the laboratory frame and in the rocket frame. Otherwise one frame could be distinguished from the other as the one with the shorter perpendicular distances. Second, the principle of relativity was employed to deduce that the speed of light must be the same in the laboratory frame as in the rocket frame—a deduction supported by the Kennedy-Thorndike experiment. The speed being the same, the fact that the light-travel path in the laboratory frame (the hypotenuses of two triangles) is longer than the simple round-trip path in the rocket frame (the altitudes of the two triangles: up 1 meter and down again) directly implies a longer *time* in the laboratory frame than in the rocket frame.

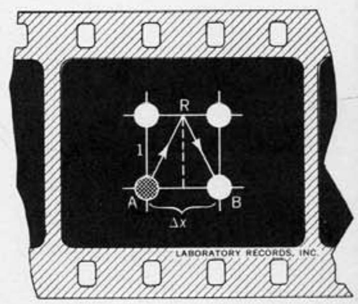


FIG. 13,A, PAGE 22

In brief, in the one elementary triangle of Fig. 13,A, are displayed the four great ideas that underlie all of special relativity: invariance of perpendicular distance, invariance of the speed of light, dependence of space *and time* coordinates upon the frame of reference, and invariance of the interval.

Is inequality of lab and rocket time lapses a paradox?

If Fig. 13,A, thus epitomizes all of special relativity in a form easy to remember, the foregoing analysis of the figure nevertheless leads to what at first sight seems to be a preposterous conclusion. How can it possibly make sense for the lapse of time between two events to be longer in the laboratory than in the rocket? Has it not already been argued that "distances at right angles to the direction of relative motion" are equal, "otherwise one frame could be distinguished from the other as the one with the shorter perpendicular distances"? What about the difference between *time* lapses in the two frames? Does not this difference give a way to differentiate physics in one frame from

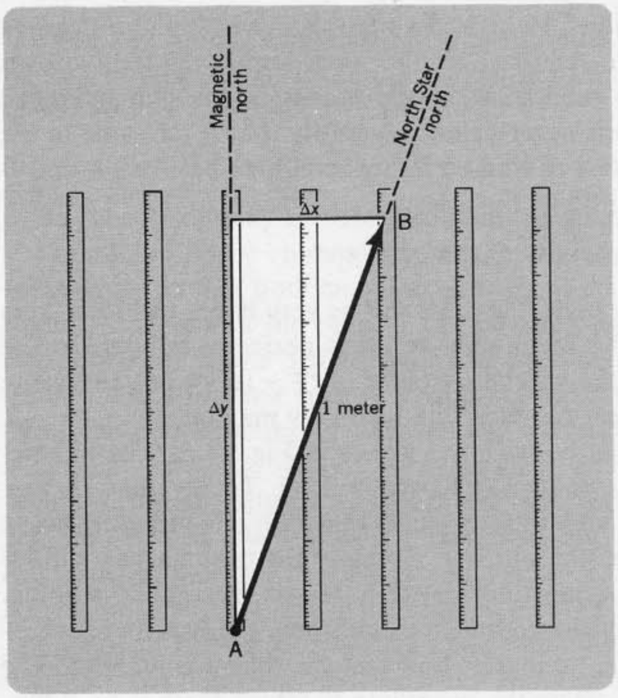


Fig. 14. The north-south separation of point B from point A (the "*northing* of B relative to A") depends upon the choice of the direction north.

physics in the other? Yet is not such a difference ruled out by the principle of relativity—the principle that one inertial reference frame is as good as another?

Relativity of time (Lorentz) compared with relativity of "northing" (Euclid)

For answers to these questions, turn back to the parable of the surveyors. Consider point B in Fig. 14. It is one meter straight north of another point, A, according to the reckoning of the Nighttime surveyor and his North Star north. Now consider the location of point B according to the Daytime surveyor and his magnetic north. Is the y separation Δy between A and B (surveying terminology: the *northing* of B relative to A) also one meter in the Daytime frame? No, Δy is *less* than one meter! How can this be? Because the altitude (Δy) of a right triangle is shorter than the hypotenuse (1 meter). Does this mean that the rules of surveying in the Daytime coordinate system are different from those in the Nighttime coordinate system? Evidently not! Similarly, there is no flaw in the construction or functioning of the laboratory clocks that makes them give longer readings for the time lapse AB. The "discrepancy" between the laboratory clocks and the rocket clock is caused instead by the character of spacetime geometry itself. That is the way the world is built! The analogy between the Lorentz geometry of spacetime and the Euclidean geometry of the surveyors' world is expanded in Table 6 (pages 28 and 29).

6. The Spacetime Diagram; World Lines

The spacetime diagram: simple way to display events

A simple way to look at the events of emission and reception of the last section is to plot the position of the event on the horizontal axis and the time of the event on the vertical axis of a *spacetime diagram* (Fig. 15). The light is *emitted* from a spark plug attached to the reference clock of the first rocket. This plug fires at the instant when this clock passes the laboratory reference