The Geometry of Spacetime

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1. Parable of the Surveyors

Daytime surveyor uses magnetic north

Once upon a time there was a Daytime surveyor who measured off the king's lands. He took his directions of north and east from a magnetic compass needle. Eastward directions from the center of the town square he measured in meters (x in meters). Northward directions were sacred and were measured in a different unit, in miles (y in miles). His records were complete and accurate and were often consulted by the Daytimers. (See Fig. 1.)

Nighttime surveyor uses North Star north

Nighttimers used the services of another surveyor. His north and east directions were based on the North Star. He too measured distances eastward from the center of the town square in meters (x' in meters) and sacred distances north in miles (y' in miles). His records were complete and accurate. Every corner of a plot appeared in his book with its two coordinates, x' and y'.

One fall a student of surveying turned up with novel openmindedness. Contrary to all previous tradition he attended both of the rival schools operated by the two leaders of surveying. At the day school he learned from one expert his method of recording the location of the gates of the town and the corners of plots of land. At night school he learned the other method. As the days and nights passed the student puzzled more and more in an attempt to find some harmonious relationship between the rival ways of recording location. He carefully compared the records of the two surveyors on the locations of the town gates relative to the center of the town square:

Table 1. Two different sets of records for the same points.

Place	*Daytime surveyor's axes oriented to magnetic north (x in meters; y in miles)*		*Nighttime surveyor's axes oriented to the North Star (x' in meters; y' in miles)*	
Town square	0	0	0	0
Gate A	x_A	y_A	x'_A	y'_A
Gate B	x_B	y_B	x'_B	y'_B
Other gates	...	...	...	...

In defiance of tradition, the student took the daring and heretical step to convert northward measurements, previously expressed always in miles, into meters by multiplication with a constant conversion factor, k. He then discovered that the quantity $[(x_A)^2 + (ky_A)^2]^{1/2}$ based on Daytime measurements of the position of gate A had exactly the same numerical value as the quantity

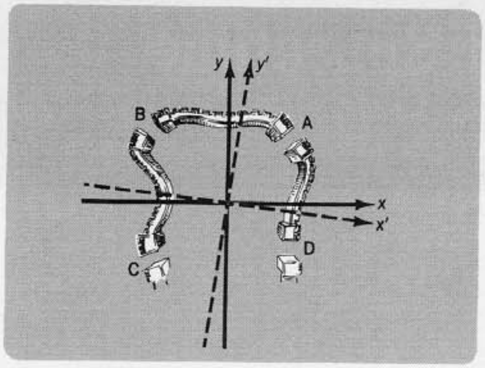

Fig. 1. The town and its gates, showing coordinate axes used by two different surveyors.

$[(x_A')^2 + (ky_A')^2]^{1/2}$ computed from the readings of the Nighttime surveyor for gate A. He tried the same comparison on the readings computed from the recorded positions of gate B, and found agreement here too. The student's excitement grew as he checked his scheme of comparison for all the other town gates and found everywhere agreement. He decided to give his discovery a name. He called the quantity

Discovery: invariance of distance

$$[(x)^2 + (ky)^2]^{1/2} \quad (1)$$

the *distance* of the point (x, y) from the center of town. He said that he had discovered the *principle of the invariance of distance;* that one gets exactly the same distances from the Daytime coordinates as from the Nighttime coordinates, despite the fact that the two sets of surveyors' numbers are quite different.

This story illustrates the naive state of physics before the discovery of *special relativity* by Einstein of Bern, Lorentz of Leiden, and Poincaré of Paris. How naive?

1. Surveyors in this mythical kingdom measured northward distances in a sacred unit, the mile, different from the unit used in measuring eastward distances. Similarly, people studying physics measured time in a sacred unit, the second, different from the unit used in measuring space. No one thought of using the same unit for both, or of what one could learn by squaring and combining space and time coordinates when both were measured in meters. The conversion factor between seconds and meters, namely the speed of light, $c = 2.997925 \times 10^8$ meters per second, was regarded as a sacred number. It was not recognized as a mere conversion factor like the factor of conversion between miles and meters—a factor that arose out of historical accidents alone, with no deeper physical significance.
2. In the parable the northbound coordinates, y and y', as recorded by the two surveyors did not differ very much because the two directions of north were separated only by the small angle of 10 degrees. At first our mythical student thought the small differences between y and y' were due to surveying error alone. Analogously, people have thought of the time between the explosion of two firecrackers as the same, by whomever observed. Only in 1905 did we learn that the time difference between the second event and the first, or "reference event," really has dif-

ferent values, t and t', for observers in different states of motion. Think of one observer standing quietly in the laboratory. The other observer zooms by in a high-speed rocket. The rocket comes in through the front entry, goes down the middle of the long corridor and out the back door. The first firecracker goes off in the corridor ("reference event") then the other ("event A"). Both observers agree that the reference event establishes the zero of time and the origin for distance measurements. The second explosion occurs, for example, 5 seconds later than the first, as measured by laboratory clocks, and 12 meters further down the corridor. Then its time coordinate is $t_A = 5$ seconds and its position coordinate is $x_A = 12$ meters. Other explosions and events also take place down the length of the corridor. The readings of the two observers can be arranged as in Table 2.

One observer uses laboratory frame

Another observer uses rocket frame

Table 2. Space and time coordinates of the same events as seen by two observers in relative motion. For simplicity the y and z coordinates are zero, and the rocket is moving in the x direction.

Event	*Coordinates as measured by observer who is*			
	standing (x *in meters*; t *in seconds*)		*moving by in rocket* (x' *in meters*; t' *in seconds*)	
Reference event	0	0	0	0
Event A	x_A	t_A	x'_A	t'_A
Event B	x_B	t_B	x'_B	t'_B
Other events	...	...	...	...

3. The mythical student's discovery of the concept of distance is matched by the Einstein-Poincaré discovery in 1905 of the idea of *interval*. The interval as calculated from the one observer's measurements

Discovery: invariance of interval

$$\text{interval} = [(ct_A)^2 - (x_A)^2]^{1/2} \tag{2}$$

agrees with the interval as calculated from the other observer's measurements

$$\text{interval} = [(ct_A')^2 - (x_A')^2]^{1/2} \tag{3}$$

even though the *separate coordinates* employed in the two calculations *do not* agree. The two observers will find different space and time coordinates for events A, B, C, . . . relative to the same reference event, but when they calculate the Einstein *intervals* between these events, their results will agree. *The invariance of the interval*—its independence from the choice of the reference frame—forces one to recognize that time cannot be separated from space. Space and time are part of the single entity, *spacetime*. The geometry of spacetime is truly four-dimensional. In one way of speaking, the "direction of the time axis" depends upon the state of motion of the observer, just as the directions of the y axes employed by the surveyors depend upon their different standards of "north."

The rest of this chapter is an elaboration of the analogy between surveying in space and relating events to one another in spacetime. Table 3 is a preview of this elaboration. To recognize the unity of space and time one follows the procedure that makes a landscape take on meaning—he looks at it from several angles. This is the reason for comparing space and time coordinates of an event in two *different* reference frames in relative motion.

Table 3. Preview: Elaboration of the parable of the surveyors.

Parable of the surveyors: geometry of space	*Analogy to physics: geometry of spacetime*
The task of the surveyor is to locate the position of a point (gate A) using one of two coordinate systems that are rotated relative to one another.	The task of the physicist is to locate the position and time of an event (firecracker explosion A) using one of two reference frames which are in motion relative to one another.
The two coordinate systems: oriented to magnetic north and to North-Star north.	The two reference frames: the laboratory frame and the rocket frame.
For convenience all surveyors agree to make position measurements with respect to a common origin (the center of the town square).	For convenience all physicists agree to make position and time measurements with respect to a common reference event (explosion of the reference firecracker).
The analysis of the surveyors' results is simplified if x and y coordinates of a point are both measured in the same units, in meters.	The analysis of the physicists' results is simplified if the x and t coordinates of an event are both measured in the same units, in meters.
The *separate* coordinates x_A and y_A of gate A *do not* have the same values respectively in two coordinate systems that are rotated relative to one another.	The *separate* coordinates x_A and t_A of event A *do not* have the same values respectively in two reference frames that are in uniform motion relative to one another.
Invariance of distance. The *distance* $(x_A{}^2 + y_A{}^2)^{1/2}$ between gate A and the town square has the same value when calculated using measurements made with respect to either of two rotated coordinate systems (x_A and y_A both measured in meters).	*Invariance of the interval.* The *interval* $(t_A{}^2 - x_A{}^2)^{1/2}$ between event A and the reference event has the same value when calculated using measurements made with respect to either of two reference frames in relative motion (x_A and t_A both measured in meters).
Euclidean transformation. Using *Euclidean* geometry, the surveyor can solve the following problem: *Given* the Nighttime coordinates x_A' and y_A' of gate A and the relative inclination of respective coordinate axes, *find* the Daytime coordinates x_A and y_A of the same gate.	*Lorentz transformation.* Using *Lorentz* geometry, the physicist can solve the following problem: *Given* the rocket coordinates x_A' and t_A' of event A and the relative velocity between rocket and laboratory frames, *find* the laboratory coordinates x_A and t_A of the same event.

Measure time in meters

The parable of the surveyors cautions us to use the same unit to measure both distance and time. So use meters for both. Time can be measured in meters. When a mirror is mounted at each end of a stick one-half meter long, a flash of light may be bounced back and forth between these two mirrors. Such a device is a *clock*. This clock may be said to "tick" each time the light flash arrives back at the first mirror. Between ticks the light flash has traveled a round-trip distance of 1 meter. Therefore the unit of time between ticks of this clock is called *1 meter of light-travel time* or more simply *1 meter of time*. (Show that 1 second is approximately equal to 3×10^8 meters of light-travel time.)

Simplify: Pick freely falling laboratory

One purpose of the physicist is to sort out simple relations between events. To do this here he might as well choose a particular reference frame with respect to which the laws of physics have a simple form. Now, the force of gravity acts on everything near the earth. Its presence complicates the laws of motion as we know them from common experience. In order to eliminate this and other complications, we will, in the next section, focus attention on a freely falling reference frame near the earth. In this reference frame no gravitational forces will be felt. Such a gravitation-free reference frame will be called an *inertial reference frame*. Special relativity deals with the classical laws of physics expressed with respect to an inertial reference frame.

The principles of special relativity are remarkably simple. They are very much simpler than the axioms of Euclid or the principles of operating an automobile. Yet both Euclid and the automobile have been mastered—perhaps with insufficient surprise—by generations of ordinary people. Some of the best minds of the twentieth century struggled with the concepts of relativity, not because nature is obscure, but simply because man finds it difficult to outgrow established ways of looking at nature. For us the battle has already been won. The concepts of relativity can now be expressed simply enough to make it easy to think correctly—thus "making the bad difficult and the good easy."† The problem of understanding relativity is no longer one of *learning* but one of *intuition*—a practiced way of seeing. When seen with this intuition, a remarkable number of otherwise incomprehensible experimental results are revealed to be perfectly natural.‡

2. The Inertial Reference Frame

Less than a month after the surrender at Appomattox ended the American Civil War (1861–65), the French author Jules Verne began writing *A Trip from the Earth to the Moon* and *A Trip around the Moon*.§ Eminent American cannon designers, so the story goes, cast a great cannon in a pit dug in the earth of Florida with the cannon muzzle pointing skyward. From this cannon is fired a 10-ton projectile containing three men and several animals. As the projectile coasts outward in unpowered flight toward the moon after leaving the cannon, its passengers walk normally inside the projectile on the side

†Einstein, in a similar connection, in a letter to the architect Le Corbusier.

‡For a comprehensive set of references to introductory literature concerning the special theory of relativity, together with several reprints of articles, see *Special Relativity Theory*, Selected Reprints, published for the American Association of Physics Teachers by the American Institute of Physics, 335 East 45th Street, New York 17, New York, 1963.

§Paperback edition published by Dover Publications, New York. Hardcover edition published in the Great Illustrated Classics Series by Dodd, Mead and Company, New York, 1962.