

Minkowski (1908) and introduce a new quantity w to measure time, a quantity defined by

$$w = (-1)^{1/2} t$$

or

$$(11) \quad \Delta w = (-1)^{1/2} \Delta t$$

Then the expression for the interval of proper distance takes the form

$$(\text{interval of proper distance})^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + (\Delta w)^2$$

The signs are now all positive. The geometry superficially appears to be that of Euclid—in four dimensions, of course, instead of three. Impressed by this formula, Minkowski wrote his famous words, “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.”† Today this union of space and time is called spacetime. Spacetime is the arena in which stars, atoms, and people live and move and have their being. Space is different for different observers. Time is different for different observers. Spacetime is the same for everyone.

Minkowski’s insight is central to the understanding of the physical world. It focuses attention on those quantities, such as interval, which are the same in all frames of reference. It brings out the relative character of quantities, such as velocity, energy, time, distance, which depend upon the choice of frame of reference.

Today we have learned not to overstate Minkowski’s argument. It is right to say that time and space are inseparable parts of a larger unity. It is wrong to say that time is identical in quality with space. Why is it wrong? Is not time measured in meters, just as distance is? Are not the x and y coordinates of the surveyor quantities of identical physical character? By analogy, are not the x and t coordinates of the spacetime diagram of the same nature as one another? How else could it be legitimate to treat these quantities on an equal footing, as in the formula $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2]^{1/2}$ for a space-like interval? Equal footing, yes; same nature, no. There is a *minus* sign in this formula that no sleight of hand can ever conjure away. *This minus sign marks the difference in character between space and time.* It does not really remove this minus sign to introduce the imaginary number $\Delta w = (-1)^{1/2} \Delta t$. It would if w were a real quantity. But w is not real. No clock ever reads $(-1)^{1/2}$ seconds, or $(-1)^{1/2}$ meters. Real clocks show real time: $\Delta t = 7$ seconds, for example. Consequently the term $-(\Delta t)^2$ is always opposite in sign to the distance term $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$. No twisting or turning can ever make the two signs the same.

The difference in sign between the time term and the space terms in the expression for the interval gives Lorentz geometry a unique feature, which is new and quite different from anything in Euclidean geometry. In Euclidean geometry it is never possible for the distance AB between two points to be zero unless all three of the quantities Δx , Δy , and Δz are simultaneously zero. In contrast, the interval AB between two events can vanish even when the

Minkowski on unity of spacetime

Difference between time and space

†A. Einstein *et al.*, *The Principle of Relativity*, (Dover Publications, New York).

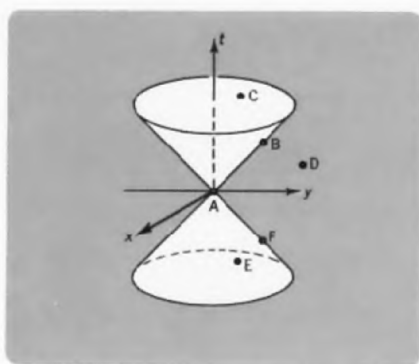


Fig. 22. Spacetime diagram showing x , y , and t coordinates of events for which $z = 0$.

separations Δx , Δy , Δz in space and Δt in time between B and A are individually quite large.

Case where interval
vanishes

Under what condition does the interval AB vanish? *The interval vanishes when the time part of the separation between A and B is identical in magnitude to the space part of the separation*

$$(12) \quad \Delta t = \pm [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}$$

What is the physical interpretation of this condition? The expression on the right is the distance between the two points. But light travels one meter of distance in one meter of light-travel time. Therefore the expression on the right also represents the time that is *needed* by light to travel the distance between A and B. On the other hand, Δt represents the time that is *available* to travel this distance. In other words, condition (12) is satisfied—and the interval AB vanishes—when a light flash starting at event A can arrive precisely in time for event B (or when a flash starting at B can arrive at A). *The interval between two events is zero when they can be connected by one light ray.*

It is interesting to map out in an appropriate diagram the location of all events B that can be connected with one given event A by a light ray. For simplicity let event A occur at the origin of the spacetime diagram. Let the coordinates x , y , z of event B be taken to have any values. Then the time coordinate of event B has either the value

$$(13) \quad t_{\text{future}} = +(x^2 + y^2 + z^2)^{1/2}$$

or the value

$$(14) \quad t_{\text{past}} = -(x^2 + y^2 + z^2)^{1/2}$$

Light cones: partitions
in spacetime

It simplifies the graphical presentation of this formula to limit attention to events B whose z coordinate is zero. Then it is appropriate to construct a spacetime diagram with two spatial coordinates x and y and the time coordinate t , as shown in Fig. 22. Every event B in this diagram that is separated from A by a zero interval ("lightlike interval") lies either on the "future light cone" of A (plus sign in Eq. 13) or on the "past light cone" of A (minus sign in Eq. 14).

In Fig. 22 consider all those events that have time coordinates 7 meters later than the zero time of flash A. These events lie on a plane 7 meters above the

xy plane, and parallel to the future light cone. This circle (circle in diagram!) is the locus of events observed at a late time. The forward light cone started at A. Similarly, the backward light cone diverging pulse of radiation at time zero.

The light cone is a feature in Euclidean geometry. In Lorentz geometry, the structure of the physical world with respect to the

1. Can a *particle* lie in the future?
2. Can a *light ray* lie on the future?
3. Can *no effect* lie outside the future?
4. Can a *particle* lie in the past?
5. Can a *light ray* lie in the past?

Now, the light cone has an existence in spacetime to describe it. The conditions, that one event is in the frame in which the causal connection

Figure 23 summarizes events of spacetime

8. The Lorentz

At heights of 10 to 15 meters, striking the nuclei of both charged and neutral (Fig. 24). In the rest frame, the average life of the meson is 2.6 $\times 10^{-8}$ seconds. The coordinates of the event of its decay are written

How do these events relate to clocks, how long

xy plane, and parallel to the xy plane. Among these events those which lie on the future light cone of A are on a circle. This circle has a radius of 7 meters. This circle (circle in the present x, y, t diagram; a sphere in a full x, y, z, t diagram!) is the locus of the pulse of radiant energy which emerged from A. Observed at a later time, the pulse has expanded to a still larger radius. Thus the forward light cone tells the history of the expanding spherical pulse that started at A. Similarly the backward light cone tells the history of a converging pulse of radiation, so perfectly focused that it collapses at the origin at time zero.

The light cone is a unique feature of Lorentz geometry; there is no such feature in Euclidean geometry. Moreover, associated with the light cone, Lorentz geometry has a characteristic of the greatest importance for the structure of the physical world. It provides the following ordering of all events with respect to their causal relationship to any chosen event A (Fig. 22).

1. Can a *particle* emitted at A affect what *is going to* happen at C? If so, C lies *in* the future light cone of A.
2. Can a *light ray* emitted at A affect what *is going to* happen at B? If so, B lies *on* the future light cone of A.
3. Can *no effect whatever* produced at A affect what happens at D? If so, D lies *outside* the light cone of A.
4. Can a *particle* emitted at E affect what *is happening* at A? If so, E lies *in* the past light cone of A.
5. Can a *light ray* emitted at F affect what *is happening* at A? If so, F lies *on* the past light cone of A.

Now, the light cone of event A—and the light cone of every other event—has an existence in spacetime quite apart from any coordinates that may be used to describe it. Therefore the possibilities mentioned in the five preceding questions, that one event will affect another event are independent of the reference frame in which this connection between events is observed. In this sense *the causal connection between two events is preserved in every reference frame.*

Figure 23 summarizes the relations between a selected event A and all other events of spacetime.

Spacetime classified into five regions relative to event A

8. The Lorentz Transformation

At heights of 10 to 30 kilometers above the earth, cosmic rays are continually striking the nuclei of oxygen and nitrogen atoms and producing π -mesons, both charged and neutral. Follow one of the π^+ -mesons on its way down (Fig. 24). In the reference frame attached to this particle ("rocket frame") the average life of the π^+ -meson is 2.55×10^{-8} seconds. In this rocket frame let the coordinates of the event of birth be $x' = 0, t' = 0$ (Fig. 25,B). Let the coordinates of the event of explosion of the π -meson (into muon plus neutrino) be written

$$x' = 0, \quad t' = \tau_\pi$$

How do these events appear to the laboratory observer? As recorded by his clocks, how long does the π -meson live from birth to death? Or how much is

Coordinates more convenient than interval for describing trip of π -meson

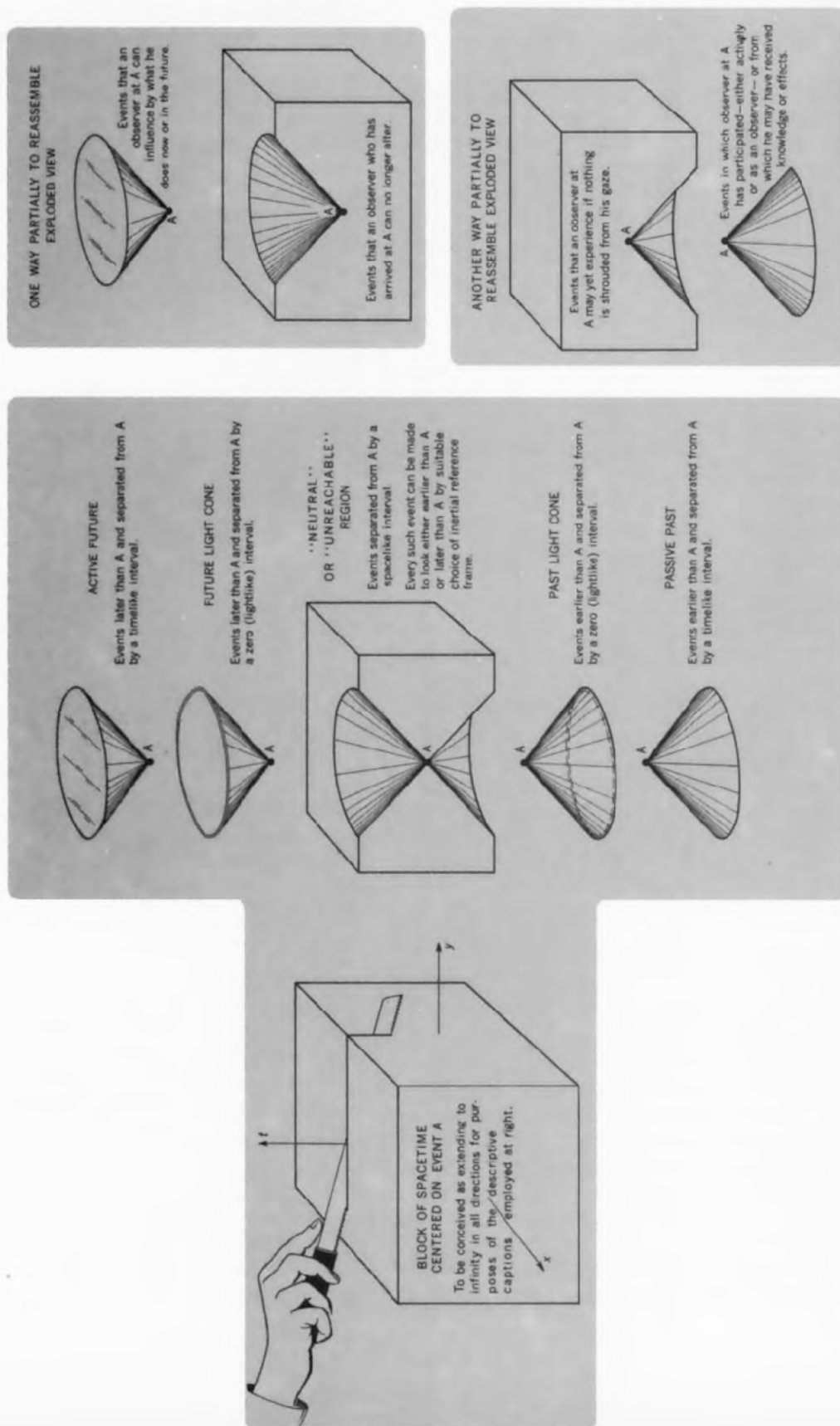
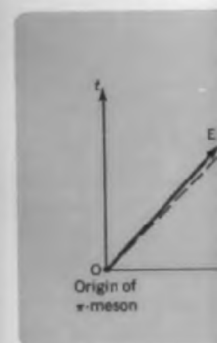


Fig. 23. Exploded view of the five regions into which the events of spacetime fall apart when classified with respect to a selected event A.

Fig. 24. Origin and distance of the lapse of labora-
laboratory distance
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origin O by know-
the coordinates x ,
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Fig. 25. Coordinates plotted on a laboratory A. Laboratory spacetime



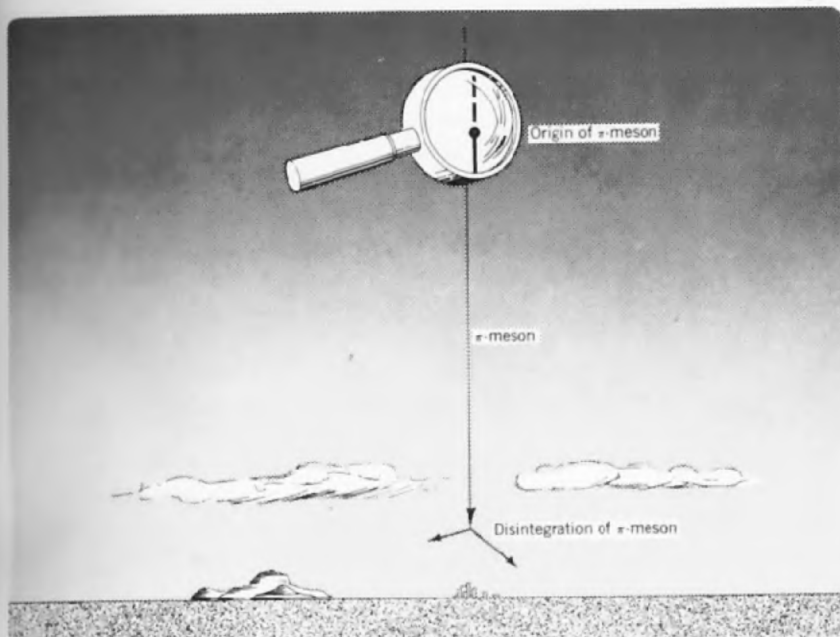


Fig. 24. Origin and disintegration of a π -meson.

the lapse of laboratory time t ? And how far does it travel? Or how much is the laboratory distance x , measured downward through the upper atmosphere from the point of formation? In a word, given an event E separated from an origin O by known coordinates x', t' in the rocket frame, how can one predict the coordinates x, t of the same event relative to the same origin in the laboratory frame (Fig. 25,A)?

This is a new kind of question. Up to now we have limited attention to the invariant interval as a way to describe the separation between two events. This interval has a value independent of the choice of reference frame; thus

$$(15) \quad (\text{spacelike interval})^2 = -(\text{timelike interval})^2 = x^2 - t^2 = (x')^2 - (t')^2$$

Fig. 25. Coordinates of the origin (point O) and disintegration (point E) of a π -meson as plotted on a laboratory and on a rocket spacetime diagram.

A. Laboratory spacetime diagram

B. Rocket spacetime diagram

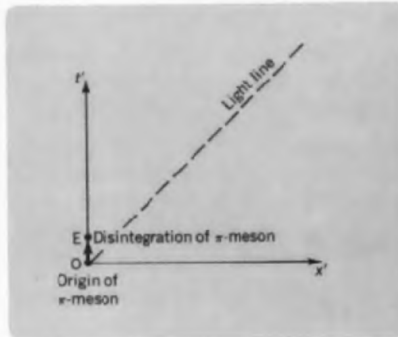
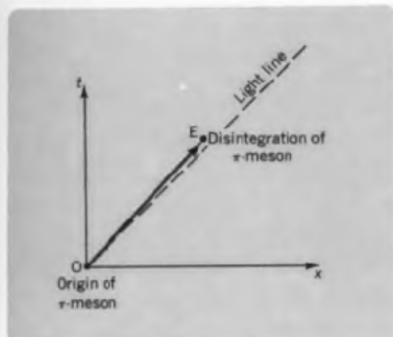


Fig. 23. Exploded view of the five regions into which the events of spacetime fall apart when classified with respect to a selected event A.

*Coordinates of event
differ from frame to
frame*

Now we focus attention on the coordinates themselves as indicators of the separation of the event E from the origin O. We do so recognizing in advance that these quantities depend upon the choice of frame of reference. In this respect the coordinates lack the universal standing that the invariant interval has as a measure of separation. But no matter. Physics has to get on with the world's work. One uses the method of describing separation that is best suited to the job in hand. On some occasions the useful fact to give about a torpedo boat is the 50-meter separation between bow and stern. On another occasion it may be much more important to know that the bow is 40 meters north of the stern and 30 meters east of it. In the present example it does not meet our needs to know that the point of disintegrative explosion of the π -meson is separated from the point of origin by an invariant interval τ of $\sim 10^{-8}$ second. We want to know the x and t coordinates individually for the separation.

*Lorentz transformation
for coordinates*

However much the (x, t) coordinates of the event E in the laboratory frame differ from the (x', t') coordinates of the same event in the rocket frame, these two sets of coordinates are related to each other by a well-defined and simple law. This law is summarized in the *Lorentz coordinate transformation*

$$\begin{aligned} x &= (1 - \beta_r^2)^{-1/2} x' + \beta_r (1 - \beta_r^2)^{-1/2} t' \\ t &= \beta_r (1 - \beta_r^2)^{-1/2} x' + (1 - \beta_r^2)^{-1/2} t' \end{aligned} \quad (16)$$

Covariant defined

where β_r is the speed of the rocket frame *relative* to the laboratory. Because of the existence of this law, the *coordinates* are said to provide a *covariant* description of the separation of events in spacetime, in contrast to the *invariant* measure of separation supplied by the *interval*. The portion "variant" of the adjective "covariant" indicates that the coordinates vary from one reference system to another. The prefix "co" implies a coordinated variation of the coordinates of all events according to the same law. Thus x' and t' differ from one event to another; and so do x and t ; but the four coefficients

$$\begin{array}{cc} (1 - \beta_r^2)^{-1/2} & \beta_r (1 - \beta_r^2)^{-1/2} \\ \beta_r (1 - \beta_r^2)^{-1/2} & (1 - \beta_r^2)^{-1/2} \end{array}$$

that connect these two sets of coordinates have values that are independent of the event under consideration.

The derivation of the Lorentz transformation formula, its use, and its points of analogy to well known features of Euclidean geometry—seen in the parable of the surveyors—form the theme of this section.

*Three ideas leading to
Lorentz transformation*

The three ideas that go into deriving the Lorentz transformation can be stated at once: (1) The coefficients in the transformation are independent of the event under consideration ("covariant transformation"). (2) The coefficients in the transformation are such as to guarantee that a point that is at rest in the rocket frame moves in the laboratory frame with a speed β_r in the x direction. (3) The coefficients are also such as to guarantee that any interval has the same value in laboratory and rocket frames.

The principles (1,2,3) have a simple application to the disintegration of the π -meson. In the laboratory reference frame this event is separated from the event of birth by coordinates (x, t) , now to be calculated in terms of the

velocity β_r of the rocket directly given by the

or

or

(17)

The timelike *interval* death in the rocket frame

Make use of Eq. 17 a

or

or

(Example: $\beta_r = (12/25)^{1/2}$; $(1 - \beta_r^2)^{-1/2}$ is 2.6 times longer life measured in the rocket frame traveled is given by the

This calculation comes from the coordinates of the point of birth.

The problem of finding the laboratory coordinates of the rocket coordinates of the event of birth, deriving the equation of the world line in a certain way in deriving the Lorentz transformation effect, we have found the Lorentz transformation

About the two remaining ideas we know nothing for an event at $x' = 0$ in the rocket frame had any finite value in the calculation. To do the calculation. To do the special event of disintegration of a point with arbitrary

velocity β_r of the rocket frame attached to the π -meson. The ratio of x to t is directly given by the velocity

$$x/t = \beta_r$$

or

$$x = \beta_r t$$

or

$$(17) \quad x^2 = \beta_r^2 t^2$$

The timelike interval defined by x and t is given by the time between birth and death in the rocket frame (where the π -meson stays always at $x' = 0$)

$$t^2 - x^2 = t'^2 - x'^2 = t'^2 - 0 = \tau_\pi^2$$

Make use of Eq. 17 and insert $\beta_r^2 t^2$ in place of x^2 in this formula. Find

$$t^2 - \beta_r^2 t^2 = t'^2 = \tau_\pi^2$$

or

$$t^2 = t'^2 / (1 - \beta_r^2) = \tau_\pi^2 / (1 - \beta_r^2)$$

or

$$t = (1 - \beta_r^2)^{-1/2} t' = (1 - \beta_r^2)^{-1/2} \tau_\pi$$

(Example: $\beta_r = (12/13)$ of the speed of light; $1 - \beta^2 = 1 - (144/169) = 25/169$; $(1 - \beta_r^2)^{-1/2} = 13/5 = 2.6$; thus π -meson life as measured in laboratory is 2.6 times longer than "proper life;" that is, 2.6 times longer than the life measured in the reference frame attached to π -meson itself.) The distance traveled is given by the velocity multiplied by the time; thus,

$$x = \beta_r t = \beta_r (1 - \beta_r^2)^{-1/2} t'$$

π -meson problem
solved

This calculation completes the original problem: to find the laboratory coordinates of the point of disintegration of the π -meson relative to its point of birth.

The problem of the π -meson was an introduction to a general problem: to find the laboratory coordinates of a given event from a knowledge of the rocket coordinates of that event. If we say that this objective is equivalent to deriving the equations of the Lorentz transformation, then we have come a certain way in deriving that transformation by the simplest of arguments. In effect, we have found two of the four coefficients in the equations of the Lorentz transformation

$$t = (1 - \beta_r^2)^{-1/2} t' + A x'$$

$$x = \beta_r (1 - \beta_r^2)^{-1/2} t' + B x'$$

About the two remaining coefficients, which we temporarily call A and B, we know nothing for an elementary reason. The π -meson was always at the point $x' = 0$ in the rocket frame. Therefore the two coefficients A and B could have had any finite values whatever without affecting the numerical results of the calculation. To determine these coefficients we turn our attention from the special event of disintegration, E, to a more general event, one which occurs at a point with arbitrary x' and t' . Once again we demand that the interval have

Derivation of Lorentz
transformation: final
details

frames. In other words,

$$Bx'^2 = t'^2 - x'^2$$

$$B^2)x'^2 = t'^2 - x'^2$$

choice of values for A and B. The values for A and B must, first, be such as to (18) vanish as it does on

x'^2 be the same on the left

A and B, from which we

Lorentz transformation of

the components x, t of a magnitude of this interval in language; they are the fast, the components of a frame provide a very special. This specialized language in another reference frame employ "space components" by itself is of no help in servers with information wish reader looking at a g that Turkish, too, uses in translating information vers on another frame of s provided by the Lorentz

n closer at hand. A Day- nary. Only by using one st readings that are made n). No such dictionary is al language of distances.

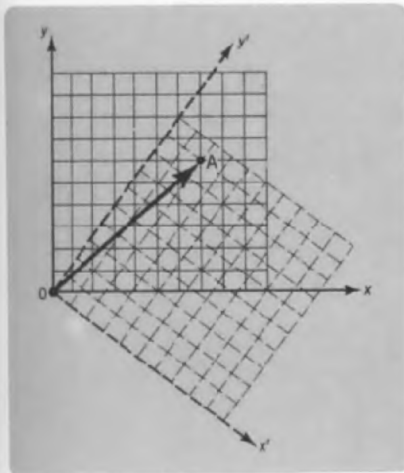


Fig. 26. The covariance approach to geometry deals with components, such as the components of the vector OA. (In contrast, the invariance approach to geometry deals with lengths, such as a length OA. Such a length has a value independent of any choice of frame of reference; in other words, it is the same whether determined by a surveyor who uses North-Star north or by a surveyor who uses magnetic north.)

The components in one frame are $(x, y) = (7, 6)$. The components in the other frame are $(x', y') = (2, 9)$, (numbers verifiable by reference to the diagram). These numbers evidently do not have the same values in the two frames of reference. Instead, they are connected by the "equations of a covariant transformation,"

$$x = (4/5)x' + (3/5)y'$$

$$y = -(3/5)x' + (4/5)y'$$

or, in the example of the vector OA,

$$7 = (4/5)2 + (3/5)9$$

$$6 = -(3/5)2 + (4/5)9$$

The special numerical values of the coefficients in the transformation equations as written above go with the special rotation illustrated in the diagram.

Evidently there is a striking contrast between the approach that focuses on invariants (distances, the universal language) and the method of description that deals with components (northward and eastward separations, different values found by the two surveyors). This contrast between invariant quantities and covariant quantities is illustrated in Fig. 26.

The student in the parable of the surveyors did only half a job, it now appears. He showed each surveyor how to translate his findings into the universal language of distance

$$(\text{distance})^2 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + (\Delta y')^2$$

However, he did not produce a dictionary that would interpret between discussions in the specialized Daytime and Nighttime languages of *components*. His achievement was useful as far as it went. But on occasion the Daytime surveyor wants to know, not merely a distance OA, but also the actual components $(\Delta x, \Delta y)$ of the separation OA. Moreover, circumstances may prevent him from directly measuring these components himself. In this event he has at his disposal only measurements of the components $(\Delta x', \Delta y')$ of the separation OA made by his colleague, the Nighttime surveyor. How is he to translate from the available numbers $(\Delta x', \Delta y')$ to the desired numbers $(\Delta x, \Delta y)$? Where is the dictionary? And what must one know in order to be able to construct the dictionary? Answer: Just as one has to know the *relative velocity* β_r of two reference frames to construct the Lorentz transformation from $(\Delta x', \Delta t')$ to $(\Delta x, \Delta t)$, so one must know the *slope* S_r of the line Oy' relative to the line Oy in order to translate from $(\Delta x', \Delta y')$ to $(\Delta x, \Delta y)$. In the example shown in Fig. 26, the slope of Oy' relative to Oy is $S_r = 3/4$. That is, for every 4 units of advance upwards along the y axis, one must travel 3 units to the right in order to arrive at the y' axis. In terms of the slope S_r the "transformation formula for rotation" is

$$\begin{aligned} \Delta x &= (1 + S_r^2)^{-1/2} \Delta x' + S_r(1 + S_r^2)^{-1/2} \Delta y' \\ \Delta y &= -S_r(1 + S_r^2)^{-1/2} \Delta x' + (1 + S_r^2)^{-1/2} \Delta y' \end{aligned} \quad (19)$$

Euclidean transformation for rotated coordinates

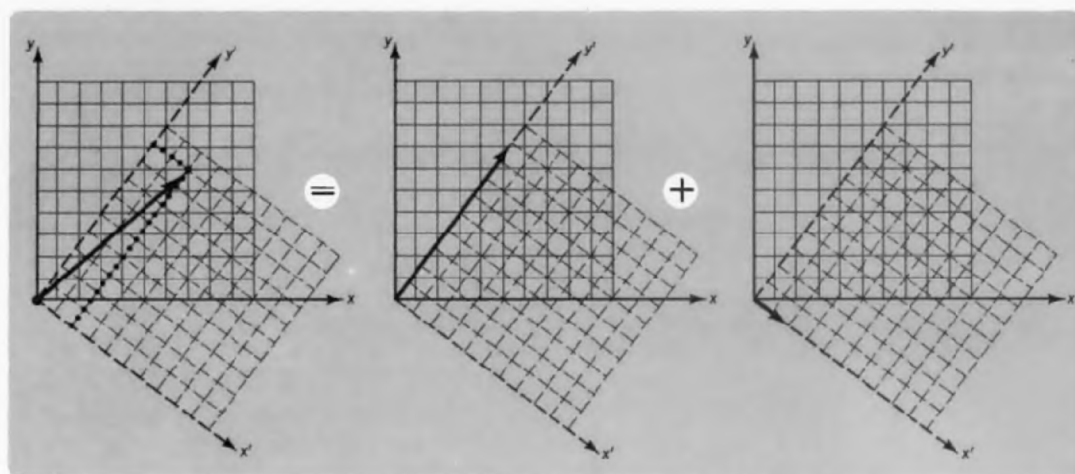


Fig. 27. Representation of the general vector as the vector sum of two vectors, which point along the y' and x' axes, respectively—a step in deriving the rotational transformation of Eqs. 19 as described in the text.

Proof:

Derivation of Euclidean transformation

1. Every arbitrary vector $(\Delta x', \Delta y')$ can be regarded (Fig. 27) as the sum of a vector $(\Delta x', 0)$ along the x' axis plus a vector $(0, \Delta y')$ along the y' axis. It is enough to establish the correctness of Eqs. 19 for these two types of vectors individually in order to confirm the correctness of Eqs. 19 in general.
2. A vector that points along the y' axis and that has length $\Delta y'$ has components along the x and y axes that stand to each other in the ratio S_r (definition of slope!); thus

$$\Delta x / \Delta y = S_r$$

or

$$(\Delta x)^2 / (\Delta y)^2 = S_r^2$$

or

$$(\Delta x)^2 = S_r^2 (\Delta y)^2$$

3. The distance from the origin to the tip of the vector has the same value in the two reference frames

$$(\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + (\Delta y')^2$$

or

$$S_r^2 (\Delta y)^2 + (\Delta y)^2 = 0 + (\Delta y')^2$$

or

$$(\Delta y)^2 = (1 + S_r^2)^{-1} (\Delta y')^2$$

or

$$\Delta y = (1 + S_r^2)^{-1/2} \Delta y'$$

and

$$\Delta x = S_r \Delta y = S_r (1 + S_r^2)^{-1/2} \Delta y'$$

Comparing these results with Eqs. 19 for a rotational transformation, we see that we have checked the two coefficients of $\Delta y'$.

4. Similarly, consider components $(\Delta x', 0)$ other in the ratio

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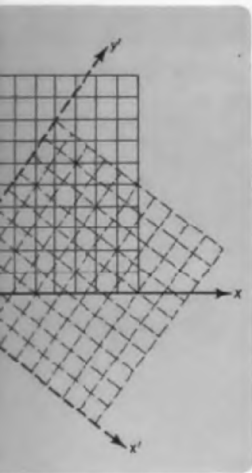
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9. The Velocity Par

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4. Similarly, consider a vector that points exclusively along the x' axis, with components $(\Delta x', 0)$. Its components along the y and x axes stand to each other in the ratio

$$\Delta y / \Delta x = -S_r$$

This information plus the invariance of the distance

$$(\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + 0$$

lead by the same type of reasoning to the result

$$\Delta x = (1 + S_r^2)^{-1/2} \Delta x'$$

$$\Delta y = -S_r (1 + S_r^2)^{-1/2} \Delta x'$$

These expressions confirm the remaining two coefficients in Eqs. 19 for a Euclidean transformation.

In summary, the covariant transformation in Euclidean geometry from $(\Delta x', \Delta y')$ to $(\Delta x, \Delta y)$ is clearly analogous to the transformation from $(\Delta x', \Delta t')$ to $(\Delta x, \Delta t)$ in the Lorentz geometry of the real physical world. The *slope* S_r of the axis of one coordinate system relative to the corresponding axis of the other system is analogous to the velocity β_r of one inertial reference frame relative to the other. The ratios between the two sides of a right triangle and its hypotenuse in Euclidean geometry

$$\frac{1}{(1 + S_r^2)^{1/2}} \quad \text{and} \quad \frac{S_r}{(1 + S_r^2)^{1/2}}$$

are replaced in Lorentz geometry by the expressions

$$\frac{1}{(1 - \beta_r^2)^{1/2}} \quad \text{and} \quad \frac{\beta_r}{(1 - \beta_r^2)^{1/2}}$$

The minus sign in the expression $(1 - \beta_r^2)^{1/2}$ contrasts with the plus sign in $(1 + S_r^2)^{1/2}$. The negative sign originates from the minus sign in the expression for the interval in Lorentz geometry.

9. The Velocity Parameter

Have we finished? We have determined how to go from a knowledge of the components of a separation in one reference frame to a calculation of the components of the separation in another reference frame. In brief, we have written down the covariant law of connection of components both for a Lorentz transformation ("transformation in x, t plane") and for a rotation ("transformation in x, y plane"). In one, the formulas contain the parameter β_r (the relative velocity); in the other, the parameter S_r (the relative slope). However, neither of these parameters provides the simplest way to describe the relation between two coordinate systems. It is desirable to replace both β_r and S_r by more natural parameters. We can find better means to describe a velocity and a rotation! *Angle* is the best measure of rotation. Similarly, a certain *velocity parameter* θ , yet to be defined, is the most convenient measure of velocity. The usefulness and meaning of this velocity parameter in describing

*Relative slope S_r
(Euclid) compared
with relative velocity
 β_r (Lorentz)*

*Additivity of angles
suggests looking for
additive velocity
parameter*

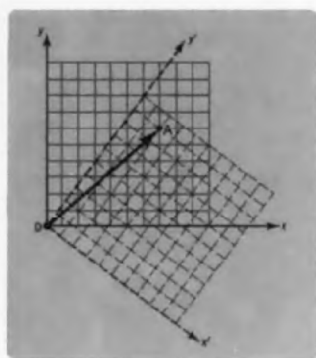


FIG. 26, PAGE 45

*Slopes in Euclidean
geometry are not
additive*

velocities will best be appreciated by asking here: Why is an angle a more convenient parameter than a slope for measuring a rotation?

And the answer is: Because *angles are additive and slopes are not*. What does this statement mean? Refer to Fig. 26. The vector OA is inclined to the y' axis. This inclination may be described by the slope S' (the number of units of distance in the x' direction per unit of distance in the y' direction). In the example this slope is

$$S' = 2/9$$

In contrast, the vector OA is inclined to the y axis by a slope

$$S = 7/6$$

Further, the y' axis is inclined to the y axis by a slope

$$S_r = 3/4$$

Question: Is the following law of addition of slope correct?

$$\left(\begin{array}{c} \text{slope of OA} \\ \text{relative to} \\ y \text{ axis} \end{array} \right) \stackrel{?}{=} \left(\begin{array}{c} \text{slope of OA} \\ \text{relative to} \\ y' \text{ axis} \end{array} \right) + \left(\begin{array}{c} \text{slope of } y' \text{ axis} \\ \text{relative to} \\ y \text{ axis} \end{array} \right)$$

Test ("experimental mathematics"):

$$\begin{aligned} (7/6) &\stackrel{?}{=} (2/9) + (3/4) \\ (42/36) &\stackrel{?}{=} (8/36) + (27/36) \\ 42 &\stackrel{?}{=} 8 + 27 = 35 \quad \text{No!} \end{aligned}$$

Conclusion: Slopes are not additive! *Question:* If slopes are not additive and S is not equal to the sum of S' and S_r , what then is the correct way to deduce the slope S from S' and S_r ? Answer:

$$\begin{aligned} \left(\begin{array}{c} \text{slope of OA} \\ \text{relative to} \\ y \text{ axis} \end{array} \right) &= S = \Delta x / \Delta y \quad (\text{by definition of slope}) \\ &= \frac{(1 + S_r^2)^{-1/2} \Delta x' + S_r (1 + S_r^2)^{-1/2} \Delta y'}{-S_r (1 + S_r^2)^{-1/2} \Delta x' + (1 + S_r^2)^{-1/2} \Delta y'} \quad (\text{from Eqs. 19}) \\ &= \frac{\Delta x' + S_r \Delta y'}{-S_r \Delta x' + \Delta y'} \quad (\text{by eliminating } (1 + S_r^2)^{-1/2} \text{ from numerator and denominator}) \\ &= \frac{(\Delta x' / \Delta y') + S_r}{-S_r (\Delta x' / \Delta y') + 1} \quad (\text{by dividing numerator and denominator by } \Delta y') \end{aligned}$$

Thus finally,

$$(20) \quad S = \frac{S' + S_r}{1 - S' S_r}$$

In other words, two slopes S' and S_r can be treated as additive only when the product $S' S_r$ in the denominator can be neglected in comparison with unity.

Fig. 28. The angle is a measure inclination between two axes—convenient because of simple law of addition: $\theta = \theta' + \theta_r$

Since slopes are not additive, the inclination of two axes is a more convenient measure this inclination. Because angle does satisfy the law of addition:

$$\left(\begin{array}{c} \text{angle of} \\ \text{inclination} \\ \text{relative to} \\ y \text{ axis} \end{array} \right) = \left(\begin{array}{c} \text{angle of} \\ \text{inclination} \\ \text{relative to} \\ y' \text{ axis} \end{array} \right) + \left(\begin{array}{c} \text{angle of} \\ \text{inclination} \\ \text{relative to} \\ y \text{ axis} \end{array} \right)$$

or

$$(21) \quad \theta = \theta' + \theta_r$$

The existence of this law of addition is the basis of the definition of angle.

What is the relation between the slope S and the angle θ ?

$$(22) \quad S_r = \tan \theta_r \quad (\text{by definition of slope})$$

Fig. 29. Relation between the angle of inclination of two corresponding axes of two coordinate systems and the angle of inclination of the axes.

Why is an angle a more rotation?

Slopes are not. What does A is inclined to the y' axis. (the number of units of the y' direction). In the ex-

a slope

pe

correct?

ope of y' axis
relative to
 y axis

pes are not additive and S correct way to deduce the

(by definition of slope)

$)^{-1/2} \Delta y'$ (from Eqs. 19)

minating $(1 + S_r^2)^{-1/2}$ from numerator and denominator)

y dividing numerator and denominator by $\Delta y'$)

as additive only when the in comparison with unity.

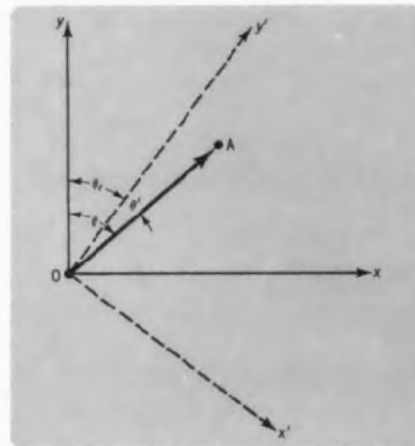


Fig. 28. The *angle* is a convenient way to measure inclination between y axis and y' axis—convenient because angles satisfy a simple law of addition: $\theta = \theta' + \theta_r$.

Since slopes are not additive and are thus not the convenient way to measure the inclination of two coordinate systems, what then is a more suitable way to measure this inclination? Answer: The *angle* between the y and y' axes. Why? Because angle does satisfy a simple law of addition (Fig. 28).

$$\left(\begin{array}{c} \text{angle of OA} \\ \text{relative to} \\ y \text{ axis} \end{array} \right) = \left(\begin{array}{c} \text{angle of OA} \\ \text{relative to} \\ y' \text{ axis} \end{array} \right) + \left(\begin{array}{c} \text{angle of } y' \text{ axis} \\ \text{relative to} \\ y \text{ axis} \end{array} \right)$$

or

$$(21) \quad \theta = \theta' + \theta_r$$

The existence of this relation makes the angle the simple measure of inclination.

What is the relation between this new measure of inclination and the old measure, the slope S_r of the y' axis relative to the y axis? Answer:

$$(22) \quad S_r = \tan \theta_r \quad (\text{from the definition of tangent in trigonometry; see Fig. 29})$$

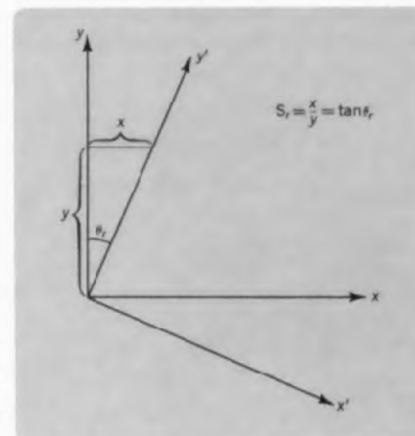


Fig. 29. Relation between relative slope S_r of corresponding axes of two Euclidean coordinate systems and the angle θ_r between these axes.

Angles ARE additive

Euclidean law of addition of slopes

Question: How does one understand the law of addition of slopes when one recognizes that a slope is the tangent of an angle? *Answer:*

$$(23) \quad \begin{aligned} \tan \theta &= \tan (\theta' + \theta_r) && \text{(additivity of angles)} \\ &= \frac{\tan \theta' + \tan \theta_r}{1 - \tan \theta' \tan \theta_r} && \text{(trigonometry)} \end{aligned}$$

or

$$S = \frac{S' + S_r}{1 - S' S_r} \quad \text{(tangents measure slopes)}$$

Comparison of the complicated law of addition of tangents, or slopes, and the simple law of addition of angles, $\theta = \theta' + \theta_r$, confirms that angles provide the simplest measure of rotations.

What is the simplest measure of velocity? Not velocity itself. Velocity itself does not satisfy a simple law of addition. What is the law of addition of velocities? Let a bullet be fired forward at a velocity β' in the rocket frame of reference (Fig. 30).

Law of addition of velocities

$$\beta' = \frac{\begin{pmatrix} \text{number of meters of advance} \\ \text{in the } x' \text{ direction for each} \\ \text{meter of advance in the read-} \\ \text{ings } t' \text{ of the rocket clocks} \end{pmatrix}}{\begin{pmatrix} \text{meter of advance in the read-} \\ \text{ings } t' \text{ of the rocket clocks} \end{pmatrix}} = (\Delta x' / \Delta t')$$

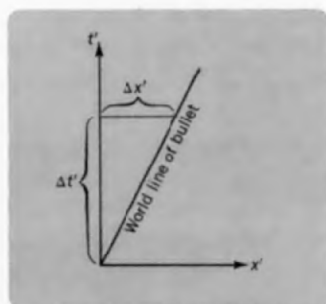


Fig. 30. World line of a bullet plotted in the rocket spacetime diagram. The bullet is fired forward with a velocity $\beta' = \Delta x' / \Delta t'$ in the rocket frame.

The rocket is moving at the velocity β_r relative to the laboratory. What is the velocity β of the *bullet* relative to the laboratory—as measured with the laboratory latticework of clocks? *Answer:* The velocity is

$$\begin{aligned} \beta &= \frac{\begin{pmatrix} \text{number of meters of advance} \\ \text{in the } x \text{ direction for each} \\ \text{meter of advance in the read-} \\ \text{ings } t \text{ of the laboratory clocks} \end{pmatrix}}{\begin{pmatrix} \text{meter of advance in the read-} \\ \text{ings } t \text{ of the laboratory clocks} \end{pmatrix}} = (\Delta x / \Delta t) \\ &= \frac{(1 - \beta_r^2)^{-1/2} \Delta x' + \beta_r (1 - \beta_r^2)^{-1/2} \Delta t'}{\beta_r (1 - \beta_r^2)^{-1/2} \Delta x' + (1 - \beta_r^2)^{-1/2} \Delta t'} && \text{(Lorentz transformation, Eqs. 16)} \\ &= \frac{\Delta x' + \beta_r \Delta t'}{\beta_r \Delta x' + \Delta t'} && \text{(by eliminating } (1 - \beta_r^2)^{-1/2} \text{ from numerator and denominator)} \\ &= \frac{(\Delta x' / \Delta t') + \beta_r}{\beta_r (\Delta x' / \Delta t') + 1} && \text{(by dividing numerator and denominator by } \Delta t') \end{aligned}$$

Thus finally,

$$(24) \quad \beta = \frac{\beta' + \beta_r}{1 + \beta' \beta_r}$$

In other words, velocities add according to the law of addition of slopes. The two velocities of accuracy) when the compared to unity (to accuracy is 1 part in 10 of velocities: The rocket is all bullet. The bullet itself rocket. What is the speed? $(3/4) + (3/4) = 1.5$ times

$$\beta = \frac{\beta' + \beta_r}{1 + \beta' \beta_r}$$

(meters of laboratory distance) (meters of laboratory distance). Thus the relativistic object can ever be proper

Considering that velocity is a new measure of velocity,

$$\left(\begin{array}{l} \text{velocity parameter} \\ \text{of bullet relative} \\ \text{to laboratory} \end{array} \right)$$

or

$$(25)$$

This parameter θ will be described rotations. The velocity angle in any diagram, as on a piece of paper and contrast, the intervals between by the Lorentz geometry of bullets and ticking clocks objects of one iota of the the additivity of the velocity but merely invites us to energy physics to see the This law of addition of velocities the law of addition of angles

What is the connection? appropriate formula is a (slope = tangent of angle)

$$(26)$$

Here “tanh” is read “hyperbolic tangent” as well as the hyperbolic

Thus finally,

$$(24) \quad \beta = \frac{\beta' + \beta_r}{1 + \beta' \beta_r} \quad (\text{law of addition of velocities})$$

In other words, velocities are not additive. *Limiting case, for low velocities only:* The two velocities β' and β_r can be treated as additive (to a certain level of accuracy) when the product $\beta' \beta_r$ in the denominator is negligibly small compared to unity (to that same level of accuracy, whether this level of accuracy is 1 part in 10 or 1 part in 10^6). *Example of lack of additivity of velocities:* The rocket is already going at $3/4$ of the speed of light when it fires a bullet. The bullet itself moves at $3/4$ of the speed of light relative to the rocket. What is the speed of the bullet relative to the laboratory? Answer: Not $(3/4) + (3/4) = 1.5$ times the speed of light, but instead

$$\beta = \frac{(3/4) + (3/4)}{1 + (3/4)(3/4)} = \frac{(3/2)}{(25/16)} = \frac{24}{25} = 0.96$$

(meters of laboratory distance per meter of travel time of light in the laboratory). Thus the relativistic law of addition of velocities (24) ensures that no object can ever be propelled at a speed as great as the speed of light.

Considering that velocities themselves are not additive, we propose to find a new measure of velocity, a "velocity parameter" θ , which *is* additive; thus,

$$\left(\begin{array}{c} \text{velocity parameter} \\ \text{of bullet relative} \\ \text{to laboratory} \end{array} \right) = \left(\begin{array}{c} \text{velocity parameter} \\ \text{of bullet relative} \\ \text{to rocket} \end{array} \right) + \left(\begin{array}{c} \text{velocity parameter} \\ \text{of rocket relative} \\ \text{to laboratory} \end{array} \right)$$

*Velocity parameter:
defined to be additive!*

or

$$(25) \quad \theta = \theta' + \theta_r$$

This parameter θ will be quite different in meaning from the angle that describes rotations. The velocity parameter cannot be represented as a simple angle in any diagram, and for a very good reason. Distances between points on a piece of paper are governed by the laws of Euclidean geometry. In contrast, the intervals between the events of the physical world are controlled by the Lorentz geometry of spacetime. But the impossibility of freezing moving bullets and ticking clocks onto a piece of paper does not deprive these lively objects of one iota of their reality. And the further impossibility of depicting the additivity of the velocity parameter θ on a page does not discourage us, but merely invites us to look at the real world of fast particles and high-energy physics to see the law of addition of velocity parameters in action. This law of addition of velocity parameters, $\theta = \theta' + \theta_r$, is every bit as real as the law of addition of angles of rotation.

What is the connection between velocity β and velocity parameter θ ? The appropriate formula is analogous to the formula for slope in terms of angle (slope = tangent of angle). It has the form

$$(26) \quad \beta = \tanh \theta$$

Here "tanh" is read "hyperbolic tangent." The hyperbolic tangent function, as well as the hyperbolic sine and cosine functions, $\sinh \theta$ and $\cosh \theta$ (with

*Velocity is hyperbolic
tangent of velocity
parameter*

mathematics. Tables of all
compilation of tables. Formal
e 8. Nevertheless, we need
cal literature. All that we
aturally enough—be found
It must correctly describe

ition

$\frac{\theta_r}{\theta_r}$ (from equation
h θ_r of definition 26)

reduce to the usual meas-
 θ must become arbitrarily
any tangent of an ordinary
provided that the angle is
degrees, there is a correc-
can here be measured in a
s, but the simplest unit is
nit the hyperbolic radian

eter and velocity be found
 $\theta = \theta$ for small velocity

enough so that $\tanh \theta$ can
accuracy. Thus, write

tangents.
n (27); thus,

$$\frac{0.01 + \tanh 0.01}{1 + 0.01(\tanh 0.01)} = \frac{0.01 + 0.01}{1 + 0.0001}$$

about the accuracy of the
value 0.02 just as we took
correction term of 0.0001
t 0.02 will depart from the
e here and now decide that
t in 10^4 . We will therefore
minator. But if we have to
y did we not make such a
correction would have been
tanh 0.01 and 0.01 can be

neglected when one is concerned to have his results correct to “only” one part in 10^4 . To this accuracy we thus finally have

$$\tanh 0.02 = \frac{0.020000}{1.0001} = 0.019998$$

(4) Now ask for the value of $\tanh 0.04$

$$\begin{aligned} \tanh 0.04 &= \tanh (0.02 + 0.02) = \frac{\tanh 0.02 + \tanh 0.02}{1 + (\tanh 0.02)(\tanh 0.02)} \\ &= \frac{2 \times 0.019998}{1 + (0.019998)^2} = 0.039980 \end{aligned}$$

The correction term in the denominator now affects the numerical value of the result by about 4 parts in 10^4 . Nevertheless the result is good to about 1 part in 10^4 . The result has been obtained by using a correct formula (Eq. 27) to combine hyperbolic tangent values, which were themselves correct to 1 part in 10^4 .

(5) We construct further entries in the hyperbolic tangent table by the same type of combinatorial procedure. Thus, from a knowledge of $\tanh 0.04$ and $\tanh 0.01$ we can calculate $\tanh 0.05 = \tanh (0.04 + 0.01)$. We go on to get $\tanh 0.1$, $\tanh 0.2$, and $\tanh 0.4$; then $\tanh 0.5 = \tanh (0.4 + 0.1)$. Similarly we calculate $\tanh 1$, $\tanh 2$, and any other values we want. In this way we find the results summarized in Fig. 31.

Two features of the velocity parameter stand out at once from Fig. 31, quite apart from any details of the numbers. First, the slope of the curve of $\tanh \theta$ versus θ goes to unity at small θ —another way of saying that the velocity, $\beta = \tanh \theta$, and the velocity parameter θ approach equality at small θ . Second, the velocity parameter θ goes to indefinitely large positive (or negative) values as the velocity, $\beta = \tanh \theta$, itself approaches plus (or minus) unity. In other

*Contrast between
velocity parameter and
ordinary angle*

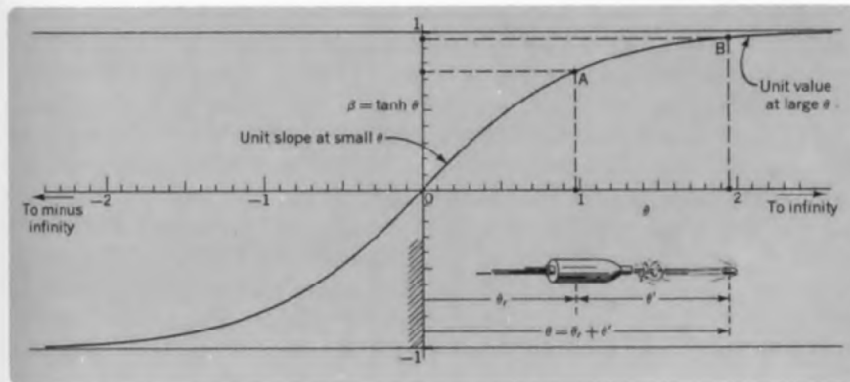


Fig. 31. Relation between velocity parameter θ and the velocity $\beta = \tanh \theta$ as determined directly from the law of addition

$$\tanh (\theta_1 + \theta_2) = \frac{\tanh \theta_1 + \tanh \theta_2}{1 + \tanh \theta_1 \tanh \theta_2}$$

as described in the text. Example: A bullet is fired at a speed $\beta' = 0.75$ from a rocket traveling at a speed $\beta_r = 0.75$. Find the speed β of the bullet relative to the laboratory. The velocity is not additive but the velocity parameter is. From the graph (point A) read off $\theta' = \theta_r = 0.973$. Add: $\theta = \theta' + \theta_r = 1.946$. For this value of the velocity parameter read off from the graph (point B) the result $\beta = 0.96$. The same result is obtained in the text in another way (p. 51).

words any values for the velocity parameter are conceivable, extending over the entire range from $\theta = -\infty$ to $\theta = +\infty$. The contrast between "hyperbolic angles" or velocity parameters, with this infinite extent of variation, and ordinary angles is evident. An ordinary angle leads to nothing new after it has increased through the finite range from 0 to 2π radians.

*Velocity parameter and
invariant speed of light*

Velocity parameters and the law of addition of velocities—what connection have these ideas with the elementary physical observations that forced on physics the spacetime point of view? The most direct connection possible: From the observations—and from what was known even in 1905 about electromagnetic waves—Einstein was led to conclude that the speed of light is the same in all inertial reference frames. In other words—to translate into the language of idealized experiments—a photon shot with the speed of light from a fast rocket travels relative to the laboratory with a speed that is *also* equal to that of light. In the language of velocity parameters, the rocket has a finite parameter θ_r ; but the photon ($\beta' = 1$) has an infinite velocity parameter ($\theta' = \infty$; Fig. 31, upper right, asymptotic limit). Add a finite number to infinity and end up with infinity for the sum $\theta = \theta' + \theta_r$. Thus the speed of the photon in the laboratory frame, $\beta = \tanh \theta = \tanh \infty = 1$, again agrees with the speed of light. We have come full circle, back to the starting idea of relativity: that the speed of light has the same value in all frames of reference.

*Simplicity of velocity
parameter*

We conclude that the velocity parameter with its simple law of addition, $\theta = \theta' + \theta_r$, is the natural way to measure velocities. Then why does one not have a direct intuitive grasp of this measure of velocity? Why is not the hyperbolic angle as familiar to every school child as the ordinary angle? The answer is simple. Everyday experience deals with angles of all sizes, large and small. Therefore no one would be so naive as to add slope $S' = 1$ (angle of 45°) to slope $S_r = 1$ (another angle of 45°) and expect to get slope $S = S' + S_r = 2$ (angle of $63^\circ 26'$. Wrong!). One knows that the correct way is to add two angles (sum: $45^\circ + 45^\circ = 90^\circ$; slope $S = \infty$). But everyday experience does not deal with velocities close to the speed of light. Motor cars, real rockets, and real bullets travel with speeds that are extremely small compared to the speed of light. Therefore it is not surprising that it took a long time to recognize the truth about spacetime physics. But now, at last, the difference that exists in nature between the law of combination of velocities (the complicated Eq. 24) and the law of combination of velocity parameters (the simple Eq. 21: $\theta = \theta' + \theta_r$) is understood. Moreover, previously perplexing observations—such as the equality of the speed of light in all reference frames—become simple to describe when one adopts the concept of the velocity parameter. In addition, this parameter—and everything that goes with it in the spacetime description of physics—are necessities. There is no substitute for these ideas for anyone who wants to look upon the structure of the physical world as that four-dimensional world really is. More and more this necessity becomes clear as electronuclear machines and high velocity particles become part of the fabric of modern civilization.

There is no way around it! The velocity parameter provides the simple way to measure speed, as the ordinary angle provides the simple way to measure inclination. Having accepted this conclusion, what profit can we draw from it in the form of a simpler way to describe a Lorentz transformation?

Ask first, by way of geometry of the xy plane coordinates in terms of

$$\Delta x$$

$$\Delta y =$$

reduce in complexity w axes in terms of the or tional transformation b

$$(1 + S_r^2)^{-1/2} = (1 + \tan^2$$

and

$$S_r(1 + S_r^2)^{-1/2}$$

Therefore the transform

(29)

and we conclude: The simplest form when the pressed as "trigonomet

Now turn to the Lo velocity

$$\Delta x =$$

$$\Delta t =$$

How does this pair of eq measure of velocity, θ ? β_r and the velocity para

Note that the coefficient by that very token are fix

(30)

and

(31)

$$\beta_r(1$$

These expressions have are well defined. For an $\tanh \theta_r$ (Fig. 31 and cor evaluate (30) and (31) v velocity parameter. The have received names of t give the functions in qu ability to find the values

Ask first, by way of orientation, the analogous question about the Euclidean geometry of the xy plane. Does the formula (Eqs. 19) for calculating one set of coordinates in terms of the other

$$\begin{aligned}\Delta x &= (1 + S_r^2)^{-1/2} \Delta x' + S_r(1 + S_r^2)^{-1/2} \Delta y' \\ \Delta y &= -S_r(1 + S_r^2)^{-1/2} \Delta x' + (1 + S_r^2)^{-1/2} \Delta y'\end{aligned}$$

Simplify Euclidean transformation using angle

reduce in complexity when one expresses the relative slope S_r of the y and y' axes in terms of the ordinary angle θ_r ? Answer: The coefficients in the rotational transformation become

$$(1 + S_r^2)^{-1/2} = (1 + \tan^2 \theta_r)^{-1/2} = \left(\frac{\cos^2 \theta_r + \sin^2 \theta_r}{\cos^2 \theta_r} \right)^{-1/2} = \left(\frac{1}{\cos^2 \theta_r} \right)^{-1/2} = \cos \theta_r$$

and

$$S_r(1 + S_r^2)^{-1/2} = \tan \theta_r \cos \theta_r = \frac{\sin \theta_r}{\cos \theta_r} \cos \theta_r = \sin \theta_r$$

Therefore the transformation equation itself takes the form

$$\begin{aligned}\Delta x &= \Delta x' \cos \theta_r + \Delta y' \sin \theta_r \\ \Delta y &= -\Delta x' \sin \theta_r + \Delta y' \cos \theta_r\end{aligned}\quad (29)$$

and we conclude: The relation between old and new coordinates takes its simplest form when the coefficients in the covariant transformation are expressed as "trigonometric," or "circular," functions of the angle of rotation.

Now turn to the Lorentz transformation written in terms of the relative velocity

$$\begin{aligned}\Delta x &= (1 - \beta_r^2)^{-1/2} \Delta x' + \beta_r(1 - \beta_r^2)^{-1/2} \Delta t' \\ \Delta t &= \beta_r(1 - \beta_r^2)^{-1/2} \Delta x' + (1 - \beta_r^2)^{-1/2} \Delta t'\end{aligned}$$

Simplify Lorentz transformation using velocity parameter

How does this pair of equations look when expressed in terms of the improved measure of velocity, θ_r ? Answer: Recall the connection between the velocity β_r and the velocity parameter

$$\beta_r = \tanh \theta_r$$

Note that the coefficients in the Lorentz transformation depend upon β_r , and by that very token are fixed by our choice of θ_r . These coefficients have the form

$$(1 - \beta_r^2)^{-1/2} = (1 - \tanh^2 \theta_r)^{-1/2} \quad (30)$$

and

$$\beta_r(1 - \beta_r^2)^{-1/2} = \tanh \theta_r (1 - \tanh^2 \theta_r)^{-1/2} \quad (31)$$

These expressions have a rather complicated appearance. Nevertheless, they are well defined. For any given value of θ_r we know how to find the value of $\tanh \theta_r$ (Fig. 31 and corresponding text). From this value of $\tanh \theta_r$ we can evaluate (30) and (31) with any desired accuracy for any given value of the velocity parameter. These two functions of θ_r have such importance that they have received names of their own in the literature on hyperbolic functions. To give the functions in question their standard names in no way decreases our ability to find the values of these functions at any time we please through our

own efforts and without reference to any treatises or tables. Therefore we accept and use the standard names hereafter:

$$\begin{aligned} (1 - \tanh^2 \theta_r)^{-1/2} &= \cosh \theta_r = \left(\begin{array}{l} \text{hyperbolic} \\ \text{cosine of } \theta_r \end{array} \right) \left\{ \begin{array}{l} \text{names;} \\ \text{nothing} \end{array} \right. \\ \tanh \theta_r (1 - \tanh^2 \theta_r)^{-1/2} &= \sinh \theta_r = \left(\begin{array}{l} \text{hyperbolic} \\ \text{sine of } \theta_r \end{array} \right) \left\{ \begin{array}{l} \text{but} \\ \text{names!} \end{array} \right. \end{aligned}$$

Using this nomenclature, we find that the equations of the Lorentz transformation take the following form

Lorentz transformation
using velocity
parameter

$$\begin{aligned} \Delta x &= \Delta x' \cosh \theta_r + \Delta t' \sinh \theta_r \\ \Delta t &= \Delta x' \sinh \theta_r + \Delta t' \cosh \theta_r \end{aligned} \quad (32)$$

and we conclude: The relation between old and new coordinates takes its simplest form when the coefficients in the transformation are expressed as hyperbolic functions of the velocity parameter θ_r of the relative motion. Moreover, expressed in terms of hyperbolic sines and cosines, the Lorentz transformation takes a form that corresponds even more closely than before to the standard trigonometric form (29) for a rotational transformation.

What can one do to grasp and feel the properties of the hyperbolic functions that appear in the Lorentz transformation? The two most interesting and important properties of these functions follow immediately from the definitions (Eqs. 30 and 31). First, the ratio of the two hyperbolic functions has the value

$$(33) \quad \sinh \theta_r / \cosh \theta_r = \tanh \theta_r$$

in complete analogy to the corresponding relation for circular functions. Second, the difference between the squares of the two hyperbolic functions is

$$(34) \quad \cosh^2 \theta_r - \sinh^2 \theta_r = \frac{1}{(1 - \tanh^2 \theta_r)} - \frac{\tanh^2 \theta_r}{(1 - \tanh^2 \theta_r)} = \frac{1 - \tanh^2 \theta_r}{1 - \tanh^2 \theta_r} = 1$$

Contrast this formula with the analogous relation for trigonometric functions

$$(35) \quad \cos^2 (\text{angle}) + \sin^2 (\text{angle}) = 1$$

Circular functions
compared with
hyperbolic functions

Equations 34 and 35 admit a simple geometrical interpretation. Plot $\sin (\text{angle})$ as the horizontal coordinate and $\cos (\text{angle})$ as the vertical coordinate in Fig. 32. Then Eq. 35 is the equation of a circle of unit radius—whence the often used term “circular functions” for the sine and the cosine. In contrast, (34) is the equation of a hyperbola (Fig. 33)—hence the word “hyperbolic functions.” The positive sign in the expression $\cos^2 + \sin^2 = 1$ has its origin in the way x components and y components of a vector are combined to obtain the square of the length of that vector. And why the minus sign in $\cosh^2 \theta - \sinh^2 \theta = 1$? Because the square of a spacetime interval is given by the square of the separation in time *diminished* by the square of the separation in space.

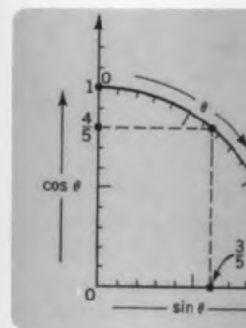


Fig. 32. Circle representing for circular functions. $(4/5)^2 + (3/5)^2 = 1$.

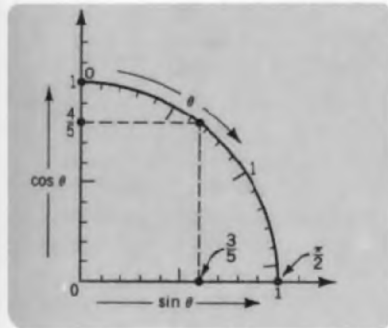
The distinction between $\cosh^2 \theta - \sinh^2 \theta = 1$ and $\cos^2 \theta + \sin^2 \theta = 1$ is the distinction between Euclidean geometry and Minkowski geometry. It confirms that the $\cosh$ and $\sinh$ functions expressed in terms of the principle of invariance $(\Delta y)^2$ from (Eqs. 29) and

$$\begin{aligned} (\text{length})^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &= (\Delta x)^2 + (\Delta y)^2 \\ &= (\Delta x)^2 + (\Delta y)^2 \\ &= (\Delta x)^2 + (\Delta y)^2 \\ &= (\Delta x)^2 + (\Delta y)^2 \end{aligned}$$

confirming the invariance of the relation

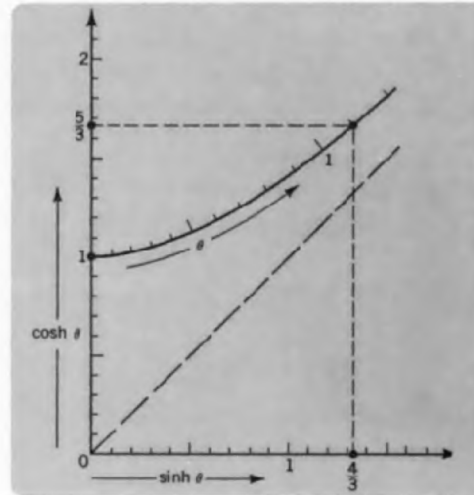
in connecting the ideas of Euclidean geometry with different orders of magnitude (length the same in both).

The connection between the two rests equally clearly on



$$\cos^2 \theta + \sin^2 \theta = 1.$$

Fig. 32. Circle representing $\cos \theta$ versus $\sin \theta$ for circular functions. Example: $(3/5)^2 + (4/5)^2 = 1$.



$$\cosh^2 \theta - \sinh^2 \theta = 1.$$

Fig. 33. Hyperbola representing $\cosh \theta$ versus $\sinh \theta$ for hyperbolic functions. Example: $(5/3)^2 - (4/3)^2 = 1$.

The distinction between the plus sign in $\cos^2 + \sin^2 = 1$ and the minus sign in $\cosh^2 \theta - \sinh^2 \theta = 1$ has to do with the contrast between the length in Euclidean geometry and the interval in Lorentz geometry. Look at this point more closely for the two kinds of geometry in turn. In Euclidean geometry reconfirm that the *covariant* transformation of *coordinates* (Eqs. 29)—now expressed in terms of circular functions rather than slope—guarantees the principle of *invariance of length*. For this purpose calculate $(\text{length})^2 = (\Delta x)^2 + (\Delta y)^2$ from (Eqs. 29) and find

$$\begin{aligned} (\text{length})^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &= (\Delta x' \cos \theta_r + \Delta y' \sin \theta_r)^2 + (-\Delta x' \sin \theta_r + \Delta y' \cos \theta_r)^2 \\ &= (\Delta x')^2 \cos^2 \theta_r + 2(\Delta x')(\Delta y') \cos \theta_r \sin \theta_r + (\Delta y')^2 \sin^2 \theta_r \\ &\quad + (\Delta x')^2 \sin^2 \theta_r - 2(\Delta x')(\Delta y') \sin \theta_r \cos \theta_r + (\Delta y')^2 \cos^2 \theta_r \\ &= [(\Delta x')^2 + (\Delta y')^2] (\cos^2 \theta_r + \sin^2 \theta_r) \\ &= (\Delta x')^2 + (\Delta y')^2 \end{aligned}$$

confirming the invariance of the expression for length. Note the importance of the relation

$$\cos^2 \theta_r + \sin^2 \theta_r = 1$$

in connecting the ideas of covariance (transformation of coordinates associated with different orientation of two coordinate systems) and invariance (length the same in both systems).

The connection between covariance and invariance in Lorentz geometry rests equally clearly on the relation

$$\cosh^2 \theta_r - \sinh^2 \theta_r = 1$$

Confirmation:
Euclidean transformation preserves distance invariant

Confirmation: Lorentz transformation preserves interval invariant

This one sees on calculating any interval whether spacelike or timelike

$$\begin{aligned}
 \left(\begin{array}{c} \text{interval of} \\ \text{proper distance} \end{array} \right)^2 &= - \left(\begin{array}{c} \text{interval of} \\ \text{proper time} \end{array} \right)^2 \\
 &= (\text{space separation})^2 - (\text{time separation})^2 \\
 &= (\Delta x)^2 - (\Delta t)^2 \\
 &= (\Delta x' \cosh \theta_r + \Delta t' \sinh \theta_r)^2 - (\Delta x' \sinh \theta_r + \Delta t' \cosh \theta_r)^2 \\
 &= (\Delta x')^2 \cosh^2 \theta_r + 2(\Delta x')(\Delta t') \cosh \theta_r \sinh \theta_r + (\Delta t')^2 \sinh^2 \theta_r \\
 &\quad - [(\Delta x')^2 \sinh^2 \theta_r + 2(\Delta x')(\Delta t') \sinh \theta_r \cosh \theta_r + (\Delta t')^2 \cosh^2 \theta_r] \\
 &= [(\Delta x')^2 - (\Delta t')^2] (\cosh^2 \theta_r - \sinh^2 \theta_r) \\
 &= (\Delta x')^2 - (\Delta t')^2
 \end{aligned}$$

Here one sees reconfirmed in the simplest way possible that a Lorentz transformation preserves the invariance of the expression for the interval.

Inverse Lorentz
transformation

The Lorentz transformation—we have now confirmed in all detail—translates from the specialized language of rocket coordinates (x' , t') to the specialized language of laboratory coordinates (x , t). Moreover, the scheme of translation is consistent at every point with the universal language of intervals (consistency of covariant description of spacetime physics with invariant description of spacetime physics). However, we need still more: The typical Turkish-English dictionary is bound together with an English-Turkish dictionary—where is the second “relativity dictionary?” How can we go backwards from a knowledge of x and t to a knowledge of x' and t' ? If one dictionary is provided by the formulas

$$\begin{aligned}
 (36) \quad x &= x' \cosh \theta_r + t' \sinh \theta_r \\
 t &= x' \sinh \theta_r + t' \cosh \theta_r
 \end{aligned}$$

what are the formulas for translation backwards from laboratory records to rocket records? *Answer:* The Lorentz transformation “inverse” to Eqs. 36 is

$$\begin{aligned}
 (37) \quad x' &= x \cosh \theta_r - t \sinh \theta_r \\
 t' &= -x \sinh \theta_r + t \cosh \theta_r
 \end{aligned}$$

Proof: Substitute these expressions for x' and t' into Eqs. 36 and verify that identities result (an English word translated into Turkish and then back into English comes out as the original word provided that the one dictionary is the true inverse of the other!).

In the following table, formal definitions of the hyperbolic functions and some of the relations that they satisfy are presented in parallel with similar definitions and relations for circular functions. In this table e is the base of the natural logarithms and has the numerical value 2.718281... The symbol i stands for the square root of minus one, so that $i^2 = -1$. The usual rules for addition and multiplication of exponents apply to exponents containing i . The angle θ is expressed in circular or hyperbolic radians (*not* degrees). The expression $4!$, for instance, means *four factorial*: $4 \times 3 \times 2 \times 1$. To understand these relations derive lines 7 to 13 from the definitions in lines 1 to 6 on each side of the table and show qualitatively how the graphs of Figs. 32 and 33 follow from these relations. Note especially the differences in *sign* between the two sides of the table.

Table 8. Circular and

Circular func

1.	$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
2.	$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$
3.	$\tan \theta = \frac{\sin \theta}{\cos \theta}$
4.	$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$
5.	$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$
6.	$\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2}{15} \theta^5 + \dots$
7.	$\sin(-\theta) = -\sin \theta$
8.	$\cos(-\theta) = \cos \theta$
9.	$\tan(-\theta) = -\tan \theta$
10.	$\boxed{\cos^2 \theta + \sin^2 \theta = 1}$
11.	$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$
12.	$\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$
13.	$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$

PC

For small θ $\sin \theta \approx \theta$
 $\tan \theta \approx \theta$

Example: $\theta = 0.1$

Poor man's recipe	$\sin$
	$\tan$
Accurate values	$\sin$
	$\tan$

Table 8. Circular and hyperbolic functions.

Circular functions	Hyperbolic functions
DEFINITIONS	
1. $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$	1. $\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$
2. $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	2. $\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2}$
3. $\tan \theta = \frac{\sin \theta}{\cos \theta}$	3. $\tanh \theta = \frac{\sinh \theta}{\cosh \theta}$
4. $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$	4. $\sinh \theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \dots$
5. $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$	5. $\cosh \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \dots$
6. $\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \dots$	6. $\tanh \theta = \theta - \frac{\theta^3}{3} + \frac{2}{15}\theta^5 - \dots$
RELATIONS	
7. $\sin(-\theta) = -\sin \theta$	7. $\sinh(-\theta) = -\sinh \theta$
8. $\cos(-\theta) = \cos \theta$	8. $\cosh(-\theta) = \cosh \theta$
9. $\tan(-\theta) = -\tan \theta$	9. $\tanh(-\theta) = -\tanh \theta$
10. $\cos^2 \theta + \sin^2 \theta = 1$	10. $\cosh^2 \theta - \sinh^2 \theta = 1$
11. $\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$	11. $\sinh(\theta_1 + \theta_2) = \sinh \theta_1 \cosh \theta_2 + \cosh \theta_1 \sinh \theta_2$
12. $\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$	12. $\cosh(\theta_1 + \theta_2) = \cosh \theta_1 \cosh \theta_2 + \sinh \theta_1 \sinh \theta_2$
13. $\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$	13. $\tanh(\theta_1 + \theta_2) = \frac{\tanh \theta_1 + \tanh \theta_2}{1 + \tanh \theta_1 \tanh \theta_2}$
POOR MAN'S QUICK RECIPES	
For small θ $\sin \theta \approx \theta$ $\tan \theta \approx \theta$	For small θ $\sinh \theta \approx \theta$ $\tanh \theta \approx \theta$
Example: $\theta = 0.1$	Example: $\theta = 0.1$
Poor man's recipe $\sin \theta \approx 0.1$ $\tan \theta \approx 0.1$	Poor man's recipe $\sinh \theta \approx 0.1$ $\tanh \theta \approx 0.1$
Accurate values $\sin \theta = 0.0998$ $\tan \theta = 0.1003$	Accurate values $\sinh \theta = 0.1002$ $\tanh \theta = 0.0997$
For large θ $\sinh \theta \approx e^{\theta}/2$ $\cosh \theta \approx e^{\theta}/2$	
Example: $\theta = 3$ $e^{\theta} \approx 20$	
Poor man's recipe $\sinh \theta \approx 10$ $\cosh \theta \approx 10$	Poor man's recipe $\sinh \theta \approx 10$ $\cosh \theta \approx 10$
Accurate values $\sinh \theta = 10.018$ $\cosh \theta = 10.068$	Accurate values $\sinh \theta = 10.018$ $\cosh \theta = 10.068$

INTRODUCTION TO THE EXERCISES OF CHAPTER 1

Important areas of current research can be analyzed very simply using the theory of relativity. This analysis depends heavily upon a physical intuition, which develops with experience. Such experience cannot be obtained in the laboratory—simple experiments in relativity are difficult and expensive because the speed of light is so great. As alternatives to simple experiments, the following exercises involve a wide range of physical consequences of the properties of spacetime. These properties of spacetime recur here over and over again in different contexts:

- paradoxes
- puzzles
- derivations
- technological applications
- estimates
- precise calculations
- philosophical difficulties

The text of the chapter has presented all formal tools necessary to answer these exercises, but intuition—a practiced way of seeing—is best developed without hurry. For this reason it will prove useful to continue to do more and more of these exercises in

relativity after one has moved on to material outside this book. Those who wish to cover the essential material in the least possible time may limit themselves to the exercises whose titles are set in boldface type in the list beginning below.

The mathematical manipulations in the exercises are very brief: only a few answers will take more than five lines to write down. On the other hand, the exercises will require some “rumination time.” Unstarred exercises should require the least time; those marked with a single asterisk are more difficult; those marked with double asterisks are suitable for graduate students in physics.

WHEELER’S FIRST MORAL PRINCIPLE. *Never make a calculation until you know the answer.* Make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every puzzle. Courage: no one else needs to know what the guess is. Therefore make it quickly, by instinct. A right guess reinforces this instinct. A wrong guess brings the refreshment of surprise. In either case life as a spacetime expert, however long, is more fun!

A. THE SPACETIME INTERVAL (Text sections 5, 6, 7)

1. Space and time—a worked example
2. Practical synchronization of clocks
3. Relations between events
4. Simultaneity
5. Temporal order of events
- *6. The expanding universe
7. Proper time in communication
8. Data-collecting and decision-making

B. THE LORENTZ TRANSFORMATION (Text sections 8 and 9)

9. Lorentz contraction—a worked example
10. Time dilation
11. Relative synchronization of clocks
12. Euclidean analogies
13. Lorentz contraction II
14. Time dilation II
15. Lorentz transformation equations with time in seconds
- *16. Derivation of the Lorentz transformation equations
- *17. Proper distance and proper time
- *18. The place where both agree

- *19. Transformation of angles
- *20. Transformation of y velocity
- **21. Transformation of velocity directions
- **22. The headlight effect

C. PUZZLES AND PARADOXES

- 23. Einstein's train paradox—a worked example
- 24. Einstein puzzler
- *25. The pole and barn paradox
- **26. Space war
- *27. The clock paradox
- *28. Things that move faster than light

D. BACKGROUND

- 29. Synchronization by a traveling clock—a worked example
- 30. Time dilation and construction of clocks
- 31. Earthbound inertial reference frames
- *32. Size of an inertial frame
- *33. The Michelson-Morley experiment
- *34. The Kennedy-Thorndike experiment
- *35. The Dicke experiment
- *36. Down with relativity!

E. APPROXIMATIONS AT LOW VELOCITY

- 37. Euclidean analogy—a worked example
- 38. The Galilean transformation
- *39. Limits of accuracy of a Galilean transformation
- *40. Collisions Newtonian and relativistic—and the domain within which the two predictions agree to one percent
- *41. Examples of the limits of Newtonian mechanics

F. SPACETIME PHYSICS: MORE OBSERVATIONS

- 42. Time dilation with μ -mesons—a worked example
- 43. Time dilation with π^+ -mesons
- *44. Aberration of starlight
- 45. Fizeau experiment
- 46. Cerenkov radiation
- *47. Deflection of starlight by the sun

G. GEOMETRIC INTERPRETATION

- 48. Geometric interpretation
- 49. The clock paradox II—a worked example

H. FREE-FOR-ALL!

- 50. Contraction or rotation?
- **51. Clock paradox III
- *52. The tilted meter stick
- *53. The meter-stick paradox
- **54. The thin man and the grid

A. THE SPACETIME INTERVAL (Text Sections 5, 6, 7)

1. Space and time—a worked example

Two events occur at the same place in the laboratory frame of reference and are separated in time by 3 seconds. (a) What is the spatial distance between these two events in a rocket frame in which the events are separated in time by 5 seconds? (b) What is the relative speed β , of the rocket and laboratory frames?

Solution: (a) The spacetime interval between these two events has the same value measured in either frame of reference

$$(\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2$$

From the statement of the problem

$$\Delta x = 0$$

$$\Delta t = 3 \text{ (seconds)} \times c \text{ (meters/second)} \\ = 9 \times 10^8 \text{ meters}$$

$$\Delta x' = \text{to be found}$$

$$\Delta t' = 5 \text{ (seconds)} \times c \text{ (meters/second)} \\ = 15 \times 10^8 \text{ meters}$$

Substitute these values into the expression for the interval

$$81 \times 10^{16} - 0 = 225 \times 10^{16} - (\Delta x')^2$$

From this equation find

$$(\Delta x')^2 = 144 \times 10^{16} \text{ meters}^2$$

or

$$\Delta x' = 12 \times 10^8 \text{ meters}$$

(b) In the *laboratory* frame the two events occur at the same place. In the *rocket* frame this laboratory “place” has moved 12×10^8 meters in 5 seconds—or in 15×10^8 meters of light-travel time. Therefore the relative speed of the two frames is

$$\Delta x' / \Delta t' = (12 \times 10^8) / (15 \times 10^8) = 4/5$$

2. Practical synchronization of clocks

You are an observer stationed near a clock with spatial coordinates $x = 6$ meters, $y = 8$ meters, and $z = 0$ meters in the laboratory frame. You wish to synchronize your clock with the one at the origin using the reference flash. Describe in detail and with numbers how to proceed.

3. Relations between events

Events A, B, and C are plotted in the laboratory spacetime diagram of Fig. 34. Answer the following questions for the pair of events A and B.

(a) Is the *interval* between the two events timelike, lightlike, or spacelike?

(b) What is the *proper time* (or *proper distance*) between the two events?

(c) Is it *possible* that one of the events *caused* the other event?

Answer the same questions for the pair of events A and C.

Answer the same questions for the pair of events C and B.

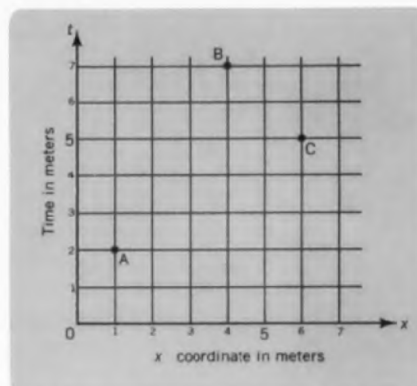


Fig. 34. What are the relations among the events A, B, and C?

4. Simultaneity

“A hits B and simultaneously one hundred million miles away C hits D.” Explain in a sentence or two how special relativity teaches us to restate or qualify this statement.

5. Temporal order of events

“Event G occurred before event H.” Prove that the *temporal order* of two events in the laboratory frame is the same as in all rocket frames if and only if the two events have either a timelike or a lightlike separation.

6.* The expanding universe

(a) A giant bomb explodes in otherwise empty space. What is the nature of the motion of one frag-

Fig. 35. Calculation of the $\Delta t_{\text{reception}}$ between arrival of server of consecutive flashes at receding emitter.

ment relative to another? Can the motion be detected? Discuss the experiment equipped with a beacon that emits light at regular, known intervals in its own frame of reference. Knowing this interval between emission, can an observer determine the velocity of the other fragment? Assume that the determination, (1) the known interval between flashes and (2) the time of arrival of consecutive flashes, is made. This is *not* equal to the time between the emission of the two flashes by the emitter; see Fig. 35.) Derive expressions for Δt and $\Delta t_{\text{reception}}$. How do these depend upon the velocity of the fragment to the fragment?

(Note: In any given time interval, events evidently travel distances. The point of explosion that is the origin of their velocities in that frame.) (b) How can observations be used to verify that the universe is expanding? Atoms in hot stars emit frequencies characteristic of the “spectral lines”. The *observed period* of the spectral line from starlight is longer than the *proper period*. From the *pattern* of spectral lines, an emitting the light can be determined. An atom can then be excited to emit light while at rest and the spectrum of any spectral line can be determined. Part (a) to describe how to determine one spectral line from starlight.

Events

plotted in the laboratory
34. Answer the following
events A and B.

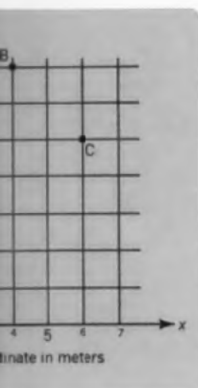
Are the two events timelike,

spacelike (or proper distance) be-

cause of the events caused the

relations for the pair of events

relations for the pair of events



relations among the events A, B,

approximately one hundred million
explain in a sentence or two
which we use to restate or qualify

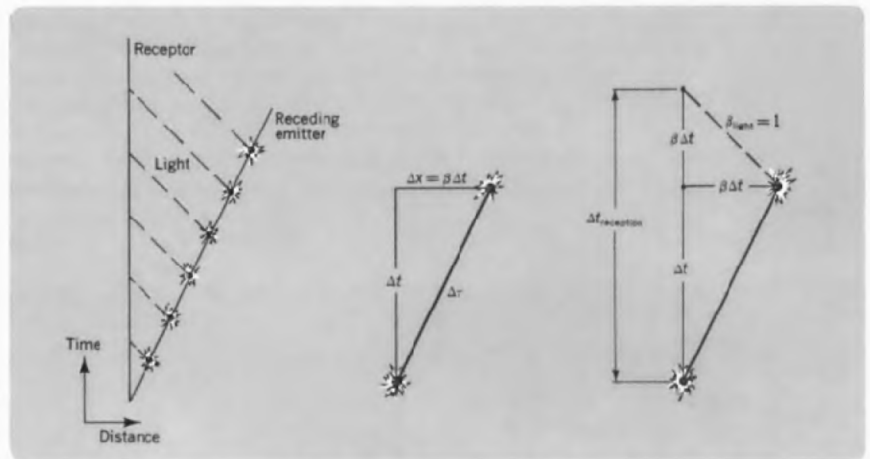
Events

more event H." Prove that the
events in the laboratory frame
frames if and only if the two
like or a lightlike separation.

Universe

explodes in otherwise empty
of the motion of one frag-

Fig. 35. Calculation of the time $\Delta t_{\text{reception}}$ between arrival at observer of consecutive flashes from receding emitter.



ment relative to another? And how can this relative motion be detected? *Discussion:* Imagine each fragment equipped with a beacon that gives off flashes of light at regular, known intervals $\Delta\tau$ of time as measured in its own frame of reference (proper time!). Knowing this interval between flashes, what method of detection can an observer on one fragment employ to determine the velocity β —relative to him—of any other fragment? Assume that he uses, in making this determination, (1) the known proper time $\Delta\tau$ between flashes and (2) the time $\Delta t_{\text{reception}}$ between the arrival of consecutive flashes at his position. (Note: This is not equal to the time Δt in his frame between the emission of the two flashes from the receding emitter; see Fig. 35.) Derive a formula for β in terms of $\Delta\tau$ and $\Delta t_{\text{reception}}$. How will the measured recession velocity depend upon the distance from one's own fragment to the fragment at which one is looking? (Note: In any given time in any given frame, fragments evidently travel distances in that frame from the point of explosion that are in direct proportion to their velocities in that frame!)

(b) How can observation of the light from stars be used to verify that the universe is expanding? *Discussion:* Atoms in hot stars give off light of different frequencies characteristic of these atoms ("spectral lines"). The observed period of the light in each spectral line from starlight can be measured on earth. From the pattern of spectral lines the kind of atom emitting the light can be identified. The same kind of atom can then be excited in the laboratory to emit light while at rest and the proper period of the light in any spectral line can be measured. Use the results of part (a) to describe how the observed period of light in one spectral line from starlight can be compared to the

proper period of light in the same spectral line from atoms at rest in the laboratory to give the velocity of recession of the star that emits the light. This observed change in period due to the velocity of the source is called the *Doppler shift*. (For a more detailed treatment see Ex. 75 of Chap. 2 and the exercises which follow it.) If the universe began in a gigantic explosion, how must the observed velocities of recession of different stars at different distances compare with one another? Slowing down during expansion—by gravitational attraction or otherwise—is to be neglected here but is considered in more complete treatments (Ex. 80).

7. Proper time in communication

A flash of light is emitted from the sun and is absorbed on the moon. "The proper time between the emission of this flash and its absorption is equal to zero." True or false? Is the proper time between the two events (emission and absorption) equal to zero if the flash is reflected back and forth between mirrors on the moon before being absorbed? (Careful!) A flash of light is emitted on earth and travels through air directly to another spot on the earth, where it is absorbed. (The speed of light in air is slightly less than c .) Is the proper time between the emission of this flash and its absorption equal to zero?

8. Data-collecting and decision-making

We have used a latticework of recording clocks to describe events. The position of an event is the position of the clock nearest to the event, and the time of an event is the time recorded on that clock. Physics deals with the study of the relations between events. If

the data-analysis center is located at the origin of the latticework of clocks, what is the lag time (in that frame) between data available for analysis at this center and data already recorded on clocks at a distance R from that center? The clock at $x = 6 \times 10^9$ meters, $y = 8 \times 10^9$ meters, and $z = 0$ meters records the passage of a meteor at 41×10^9 meters of time.

B. THE LORENTZ TRANSFORMATION (Text Sections 8 and 9)

9. Lorentz contraction—a worked example

A meter stick is attached to a rocket. The meter stick is observed from the laboratory frame of reference (laboratory framework of rods and clocks). In what way will the findings of the laboratory observer about the length of the meter stick contrast with those predicted by pre-relativity physics? We break this broad question down into four parts:

(a) How can this question about *length* be translated into a question about the separation of two events? Remarks: Each end of the meter stick traces out a world line through spacetime. But one world line is an infinite succession of events. So how is one going to pick out, in a reasonable way, exactly two events that will give the desired information about the apparent length of the meter stick?

Solution: Select the following two events for attention. A: One end of the meter stick flashes past a laboratory clock just as that clock reads noontime. B: The other end of the meter stick flashes past another laboratory clock when it too reads noontime. *Discussion:* One must measure the location of both ends of the moving meter stick at the same time in the laboratory frame. Otherwise there would not be a well defined pair of laboratory points between which to carry out the length measurement. The two events are thus simultaneous in the laboratory frame of reference ($\Delta t = 0$). They may or may not be simultaneous in the rocket frame ($\Delta t'$ may or may not be zero). No matter! The meter stick is at rest in the rocket frame. In that frame the two ends may be located at leisure.

(b) When the meter stick points along the x axis (direction of motion) of the rocket so that the separation of the two ends in the rocket frame is $\Delta x' = 1$ meter, what length is observed in the laboratory frame?

The clock at $x = 3 \times 10^9$ meters, $y = 4 \times 10^9$ meters, and $z = 0$ meters records the passage of the same meteor at 47×10^9 meters of time. The observers in the data-analysis center require 3 seconds to take evasive action. If the data above are sent to them by light flash and are displayed instantly upon arrival, will they have time to protect themselves?

Solution: The length is the separation in space of the two events A and B in the laboratory frame

$$(38) \quad \Delta x = \Delta x' / \cosh \theta_r = \Delta x' (1 - \beta_r^2)^{1/2}$$

This length is less than one meter. The shortening is called the *Lorentz contraction*. *Discussion:* The Lorentz transformation (Eqs. 37) connects separations in the laboratory frame with separations in the rocket frame by the equations

$$(39) \quad \begin{aligned} \Delta x' &= \Delta x \cosh \theta_r - \Delta t \sinh \theta_r \\ \Delta t' &= -\Delta x \sinh \theta_r + \Delta t \cosh \theta_r \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned}$$

The two events are simultaneous in the laboratory frame ($\Delta t = 0$). Therefore $\Delta x' = \Delta x \cosh \theta_r$, from which the answer follows. Note that $\Delta t'$ is not equal to zero; that is, the events A and B are not simultaneous as recorded in the rocket frame. This difference in time between events at the two ends of the meter stick raises no questions in the minds of the rocket workers as to the length of the meter stick. To them it is at rest and it is one meter long. Neither are they troubled that the laboratory observers record the length as shortened ("Lorentz contracted"). "Why not?" they say. "The laboratory observers marked down the positions of the two ends of the meter stick at times, t_A' and t_B' , that we know to be different. How could they help but get a length different from 1 meter?"

(c) When the meter stick points along the y axis (perpendicular to the direction of motion) of the rocket frame, so that the separation of the two ends in the rocket frame is $\Delta y' = 1$ meter, what length is observed in the laboratory frame?

Solution: The length is the separation in space of the two events A and B in the laboratory frame

$$\Delta y = \Delta y'$$

This length is 1 meter. The dimensions perpendicular to the direction of motion are the same in both frames. *Discussion:* Note that the two events are simultaneous in the laboratory frame ($\Delta t = 0$), but not simultaneous in the rocket frame ($\Delta t' \neq 0$; see Eqs. 39). The rocket workers should agree with the laboratory observer's meter stick.

(d) Reconsider the question: How can one possibly accept the notion that the meter stick appears to be shorter in the laboratory frame? If this conclusion is correct, how can one not have a way to distinguish between the two frames (where meter sticks are at rest)? The answer lies from the physics in the laboratory frame. The meter sticks are recorded in the laboratory frame. It does not matter how one reasons about the foundation principle of relativity. It states that one cannot distinguish between the two frames by any experiment. The physics in the two frames is the same. The striking difference between the two frames is in the way they are recorded.

Solution: Yes, there is a difference. The x dimensions recorded in the laboratory frame are the same as in the rocket frame. There is no difference between the two frames. A meter stick is at rest in the rocket frame and that is the direction of motion, is recorded in the laboratory frame. However, the rocket workers rest in the laboratory frame. The direction of motion is recorded by the rocket workers. The posterous story is this! The rocket workers and defy all this relativistic nonsense. A rocket meter stick is recorded in the laboratory as a half meter. That a length of a half meter is recorded as a full meter in the rocket dimensions is a different story. The dimensions—along the direction of motion—are as different as the dimensions perpendicular to the direction of motion in the two frames. I would have no objection whether I was in the laboratory or the rocket frame. Principle of relativity! Reply: Perhaps a different story. Lorentz disturbing at a distance. I have had so little experience with relativity at really high velocities. I am happier with the principle of relativity.

meters, $y = 4 \times 10^9$ meters, is the passage of the same amount of time. The observers in the rocket require 3 seconds to take the message above and are sent to them by a message relayed instantly upon arrival, to protect themselves?

8 and 9)

is the separation in space of events A and B in the laboratory frame

$$\Delta x = \Delta x' (1 - \beta^2)^{1/2}$$

is one meter. The shortening of the meter stick is called *contraction*. Discussion: The Lorentz transformation (Eqs. 37) connects separations in the laboratory frame with separations in the rocket frame

$$\Delta x \cosh \theta_r - \Delta t \sinh \theta_r$$

$$\Delta x \sinh \theta_r + \Delta t \cosh \theta_r$$

simultaneous in the laboratory frame. Therefore $\Delta x' = \Delta x \cosh \theta_r$, which follows. Note that $\Delta t'$ is not zero, that is, the events A and B are not simultaneous in the rocket frame. The Lorentz transformation between events at the two ends of the meter stick raises no questions in the laboratory frame as to the length of the meter stick. However, the rocket workers as to the length of the meter stick. It is at rest and it is one meter long. They are troubled that the Lorentz transformation record the length as shortened (contracted). "Why not?" they ask. The Lorentz observers marked down the Lorentz coordinates of the two ends of the meter stick at the same time in the laboratory frame. We know to be different. The Lorentz workers get a length different

from the Lorentz points along the y axis (the direction of motion) of the Lorentz separation of the two ends of the meter stick. $\Delta y' = 1$ meter, what length is the Lorentz meter stick in the laboratory frame?

is the separation in space of events A and B in the laboratory frame

$$\Delta y = \Delta y'$$

This length is 1 meter. There is no shortening of dimensions perpendicular to the direction of motion. Discussion: Note that the two events are now simultaneous not only in the laboratory frame ($\Delta t = 0$), but also in the rocket frame ($\Delta t' = 0$; see Eqs. 39). Thus it is not surprising to the rocket workers that the laboratory observers should agree with them about the length of the meter stick.

(d) Reconsider the conclusion of part b. How can one possibly accept the result that a rocket meter stick appears to be shorter than one meter to laboratory observers? If this conclusion were true, would we not have a way to distinguish the physics in the rocket frame (where meter sticks have their standard length) from the physics in the laboratory (where the same meter sticks are recorded as shortened)? And if so, does not the reasoning of relativity destroy the very foundation principle of relativity? This principle states that one cannot distinguish between one inertial frame and another by any difference between the physics in the two frames. Have we not found a most striking difference between the physics in the two frames?

Solution: Yes, there is a difference between the x dimensions recorded in the two frames; but there is no difference between the *physics* in the two frames. A meter stick that is at rest relative to the rocket and that points along the direction of motion, is recorded as shorter than 1 meter in the laboratory. However, a meter stick that is at rest in the *laboratory* and that is parallel to the direction of motion is recorded as also shortened by the *rocket* workers. Objection: What preposterous story is this! I will stick to simple logic and defy all this relativity nonsense. You say that a rocket meter stick may be recorded in the laboratory as a half meter. Then you must agree that a length of a half meter in the laboratory is recorded as a full meter in the rocket frame. So rocket dimensions are longer than laboratory dimensions—along the direction of motion. Physics is as different as it could well be between the two frames. I would have no trouble at all telling whether I was in the laboratory frame or the rocket frame. Principle of relativity! What delusion! Reply: Perhaps all of us find Einstein and Lorentz disturbing at a first encounter because we have had so little experience with objects moving at really high velocities. Perhaps you will feel happier with the principle of relativity if you see

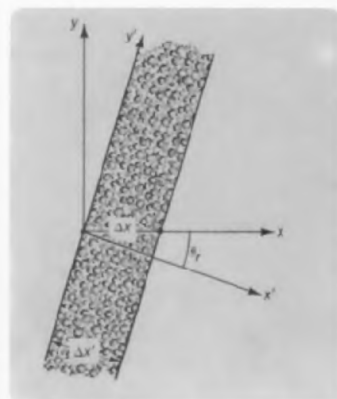


Fig. 36. A pasture extends for a greater distance in the x direction than in the x' direction.

its analog in Euclidean geometry. Of course there are some differences between the formula $(\Delta L)^2 = (\Delta x)^2 + (\Delta y)^2$ in Euclidean geometry and $(\Delta \tau)^2 = (\Delta t)^2 - (\Delta x)^2$ in Lorentz geometry. However, the question whether distances are different in two frames clearly worries you more than the question whether the distance in the new frame is less than in the old frame (Lorentz contraction in Lorentz geometry) or greater (length increase in Euclidean geometry). So look at Fig. 36. A pasture that extends for the distance $\Delta x'$ in the x' direction evidently extends for a greater distance in the x direction

$$(40) \quad \Delta x = \Delta x' / \cos \theta_r$$

On the other hand, look now at Fig. 37 (see Ex. 48 for space and time analogs of Figs. 36 and 37). Here there is another field, which extends for the distance Δx in the x direction. However, its extension in the x' direction is greater

$$(41) \quad \Delta x' = \Delta x / \cos \theta_r$$

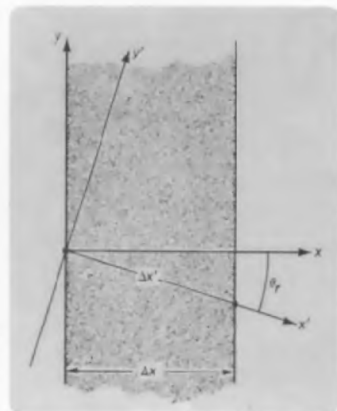


Fig. 37. Another field extends for a greater distance in the x' direction than in the x direction.

Surely you accept these results. You do not even worry about any inconsistency between formulas 40 and 41. You know as well as anyone that the Δx 's in the two formulas refer to *different measurements on different fields*. So perhaps you will be willing to believe that the length of a meter stick that is at rest relative to a rocket will be recorded as less than a meter in the laboratory, whereas a meter stick that is at rest in the laboratory will be less than a meter to the recorders on the rocket. Response: I now agree that there is no logical inconsistency in what you have been telling me. But perhaps you will go a step further and really prove to me what you have just now said about a laboratory meter stick being recorded as less than a meter in the rocket frame. Answer: Solve the Lorentz transformation equations (Eqs. 39) for the coordinates in the laboratory frame in terms of the coordinates in the rocket frame; or merely interchange the role of the primed and unprimed coordinates in those equations, and reverse the sign of the velocity; or look up Eqs. 36, inverse to Eqs. 39; in any case, write down the relations

$$(42) \quad \begin{aligned} \Delta x &= \Delta x' \cosh \theta_r + \Delta t' \sinh \theta_r \\ \Delta t &= \Delta x' \sinh \theta_r + \Delta t' \cosh \theta_r \\ \Delta y &= \Delta y' \\ \Delta z &= \Delta z' \end{aligned}$$

Our new meter stick is at rest in the laboratory frame. It is moving as viewed from the rocket frame. Consequently a determination of its length in the rocket frame requires us to have in the rocket frame two fiducial points: the locations of the two ends of the meter stick at the same rocket time. Thus $\Delta t' = 0$. From the first of Eqs. 42 we find immediately

$$(43) \quad \Delta x' = \Delta x / \cosh \theta_r = \Delta x (1 - \beta^2)^{1/2}$$

The length recorded in the rocket frame is less than one meter when the meter stick is at rest in the laboratory—as was to be shown.

10. Time dilation

A clock is carried by a rocket (Fig. 38). The clock is observed from the laboratory frame of reference (laboratory latticework of rods and clocks). In what way will the findings of the laboratory observer about the time readings of the traveling clock contrast with those predicted by pre-relativity physics? Break this question down into four parts.

(a) How can this question about time lapse be translated into a question about the separation of two events?

(b) Let the rocket clock read one meter of light-travel time between the two events chosen in part a, so that the lapse of time recorded in the rocket frame is $\Delta t' = 1$ meter. Show that the time lapse observed in the laboratory frame is given by the expression

$$(44) \quad \Delta t = \Delta t' \cosh \theta_r = \Delta t' / (1 - \beta^2)^{1/2}$$

This time lapse is *more* than one meter of light-travel time. Such lengthening is called *time dilation* ("to dilate" means "to stretch").

(c) How can one possibly accept the conclusion of part b that one meter of rocket time appears longer than one meter to laboratory observers? Does not this result give one a way to distinguish the physics in the rocket frame (where clocks run at their standard rate) from the physics in the laboratory frame (where the same clocks are recorded as running slow)? Therefore does not this reasoning violate the *principle of relativity* (Sect. 3) on which rests the whole theory of relativity?

(d) Go one step further and show that one meter of time as recorded by a clock carried in the laboratory frame ($\Delta t = 1$ meter) is recorded as more than one meter of time by observers in the rocket frame, according to the formula

$$(45) \quad \Delta t' = \Delta t \cosh \theta_r = \Delta t / (1 - \beta^2)^{1/2}$$

In what way does this result verify the symmetry between laboratory and rocket frames required by the principle of relativity?

11. Relative synchron

(a) Show that if two events occur *simultaneously in a frame but not at the same place* in the laboratory frame, they will be observed in any moving rocket frame as occurring at different times. Show that whether two events are simultaneous is a *relativity of simultaneity*.

(b) Two events occur at the same x coordinate in the laboratory frame but are separated by the y and z coordinates. Show that these two events are simultaneous in the rocket frame.

(c) Use the Lorentz transformation to show that at $t = 0$ in the laboratory frame, clocks along the positive x axis in the rocket frame will be set behind those in the laboratory frame. Show that clocks farther from the origin in the rocket frame appear to be set ahead of those in the laboratory frame, with clocks farther ahead, according to

$$(46) \quad t' = -x \sinh \theta_r$$

(d) Use the inverse Lorentz transformation to show that at $t' = 0$ in the rocket frame, clocks along the positive x axis in the laboratory frame appear to be set ahead of those in the rocket frame, with clocks farther ahead; and that clocks along the negative x axis in the laboratory frame appear to be set behind those in the rocket frame, with clocks farther behind, according to

$$(47) \quad t = +x' \sinh \theta_r$$

The fact that neither of these results agrees with the prediction of pre-relativity physics that the reference time is zero on all clocks simultaneously is called the *relativity of simultaneity*.

(e) The difference in simultaneity between the two frames that might be used to explain the results would violate the principle of relativity if an observer in *either* frame could detect the direction of motion. Then physical measurements of the time between clocks will give results in

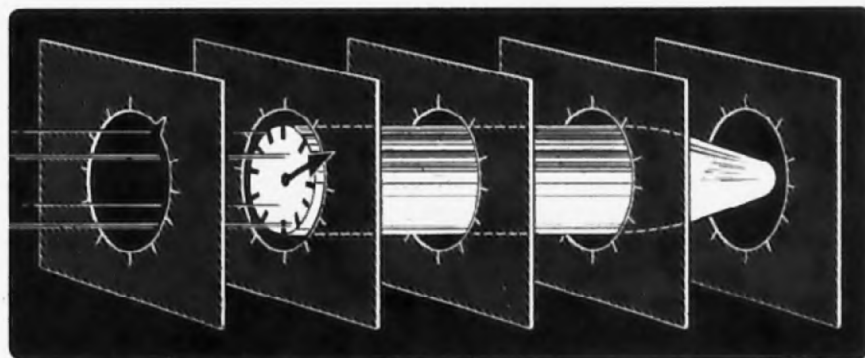


Fig. 38. A method for comparing several laboratory clocks with one rocket clock.

11. Relative synchronization of clocks

(a) Show that if two events occur simultaneously *and at the same place* in the laboratory frame they will occur simultaneously in all rocket frames. Show that if two events occur simultaneously in the laboratory frame but *not at the same position* on the x axis of the laboratory frame, they will *not* be simultaneous as observed in any moving rocket frame. The fact that observers in relative motion do not always agree whether two events are simultaneous is called the *relativity of simultaneity*.

(b) Two events occur simultaneously and at the same x coordinate in the laboratory frame, but are separated by the y and z coordinates Δy and Δz . Show that these two events are also simultaneous in the rocket frame.

(c) Use the Lorentz transformation equation to show that at $t = 0$ in the laboratory frame the clocks along the positive x axis in the rocket frame appear to be set behind those in the laboratory frame, with clocks farther from the origin set farther behind; and that clocks along the negative x axis in the rocket frame appear to be set ahead of those in the laboratory frame, with clocks farther from the origin set farther ahead, according to the equation

$$(46) \quad t' = -x \sinh \theta_r = -x \beta_r / (1 - \beta_r^2)^{1/2}$$

(d) Use the inverse Lorentz transformation equation to show that at $t' = 0$ in the rocket frame the clocks along the positive x axis in the laboratory frame appear to be set ahead of those in the rocket frame, with clocks farther from the origin set farther ahead; and that clocks along the negative x axis in the laboratory frame appear to be set behind those in the rocket frame, with clocks farther from the origin set farther behind, according to the equation

$$(47) \quad t = +x' \sinh \theta_r = +x' \beta_r / (1 - \beta_r^2)^{1/2}$$

The fact that neither of two observers in relative motion agrees that the reference event and the reading of zero time on all clocks of the *other* frame occur simultaneously is called the *relative synchronization* of clocks.

(e) The difference in sign between the equations in parts c and d seems to imply an asymmetry between frames that might be used to tell them apart—which would violate the principle of relativity. Show that if an observer in *either* frame chooses his positive x axis to lie in the direction of motion of the other frame, then physical measurements on the synchronization of clocks will give results in the two frames which are

indistinguishable. In other words, the two frames themselves are indistinguishable using this method. The difference in sign between the above equations is due to an arbitrary—and asymmetric—choice of a *common* direction for both positive x axes.

(f) The foregoing results are sometimes summarized by stating that a “rocket observer sees the laboratory clocks to be out of synchronism with one another.” Explain what is wrong with this way of stating the matter. Show that a single rocket observer is not enough to make the required measurements. What is a sharp, clean, legalistically correct, and clear (even if considerably longer!) way to state the same result?

12. Euclidean analogies

(a) A straight rod lies in the xy plane of a Euclidean coordinate system. Draw a diagram showing the rod in the xy plane; label the projections of this rod on the x , y and x' , y' axes. Spell out an explicit analogy between the x components of the length of this rod as measured in two rotated Euclidean coordinate systems and the different lengths of a moving rod observed in the laboratory frame and in the rocket frame in which the rod is at rest.

(b) Spell out an explicit analogy between time dilation and the y components of length of the rod of part a as observed in two rotated Euclidean coordinate systems. What are the Euclidean and Lorentz invariants?

(c) Spell out an explicit analogy between the relative synchronization of clocks and the case of two rotated Euclidean coordinate systems in which points on the positive x axis of *one* coordinate system have, say, a negative y coordinate in the *other* coordinate system (more negative for points farther from the common origin).

13. Lorentz contraction II

A meter stick lies along the x' axis and at rest in the rocket frame. Show that an observer in the laboratory frame will conclude that the meter stick has undergone Lorentz contraction if he measures how long it takes the meter stick to pass one of his clocks and multiplies this result by the relative velocity of the two frames.

14. Time dilation II

Two events occur at the same place but at different times in the rocket frame. Show that an observer in the laboratory frame will conclude that the time between

rocket (Fig. 38). The clock is in the laboratory frame of reference (of rods and clocks). In what way does the laboratory observer about traveling clock contrast with relativity physics? Break this apart.

Question about time lapse between the separation of two

clock read one meter of light-travel time between two events chosen in part a, recorded in the rocket frame at the time lapse observed in the laboratory frame given by the expression

$$\theta_r = \Delta t' / (1 - \beta^2)^{1/2}$$

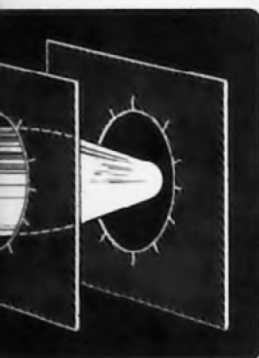
can one meter of light-travel time is called *time dilation* (“to dilate”).

Why accept the conclusion of relativity that rocket time appears longer than laboratory time? Does not this distinguish the physics in the laboratory frame (where the clocks run at their standard rate) from the rocket frame (where the clocks are running slow)? Therefore the principle of relativity is violated; the whole theory of relativity is wrong.

and show that one meter of light-travel time carried in the laboratory frame is recorded as more than one meter in the rocket frame, according to the expression

$$\theta_r = \Delta t / (1 - \beta^2)^{1/2}$$

result verify the symmetry between the two frames required by the principle of relativity.



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l (Section 5): (4) $t^2 - x^2 =$
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C. Do the results obtained in
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and proper time

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frame can be found in which
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lie along these lines. Draw
field lines due to an isolated
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ame and (b) the laboratory
follow concerning the forces
y frame, on stationary test
harge moving in that frame?

n of y velocity

uniform speed $\beta_r = \Delta y' / \Delta t'$
rocket frame. Transform the

components of y and t displacements using the
Lorentz transformation equations. Show that the
 x component and the y component of the velocity of
this particle in the laboratory frame are given by the
expressions

$$(49) \quad \begin{aligned} \beta_x &= \tanh \theta_r \\ \beta_y &= \beta_y' / \cosh \theta_r \end{aligned}$$

21.** Transformation of velocity directions

A particle moves with a velocity β' in the $x'y'$ plane
of the rocket frame in a direction that makes an angle
 ϕ' with the x' axis. Find the angle that the velocity
vector of this particle makes with the x axis of the
laboratory frame. (Hint: Transform displacements
rather than velocities.) Why does this angle differ from
that found in Ex. 19? Contrast the two results when
the relative velocity between the rocket and laboratory
frames is very great.

C. PUZZLES AND PARADOXES

23. Einstein's train paradox— a worked example

Three men (A, O, and B) are riding on a train at a
velocity β_r close to the value one. A is in front, O is at
the middle, and B is at the rear (Fig. 39). A fourth
man O' is standing beside the rails. At the very instant
O passes O' it happens that two flash-bulb signals
coming from A and B reach O and O'. Who emitted
the signal first? Using *only* the fact that the speed of
light is finite and independent of the source velocity,
show that O and O' give different answers to this
question. Having answered this qualitative question,
evaluate the quantitative difference between the times
of emission of the flashes from A and B as observed in
the frame of reference of the train (Δt_{BA}) and the
frame of reference of O' ($\Delta t_{BA}'$), either by the Lorentz
transformation or by other means.

Fig. 39. Did rider A or rider B emit his flash first?



22.** The headlight effect

A flash of light is emitted at an angle ϕ' with
respect to the x' axis of the rocket frame. Show that
the angle ϕ that the direction of this flash makes with
respect to the x axis of the laboratory frame is given
by the equation

$$(50) \quad \cos \phi = \frac{\cos \phi' + \beta_r}{1 + \beta_r \cos \phi'}$$

Show that your answer to the previous exercise gives
the same result when the velocity β' is given the value
one. Now consider a particle at rest in the rocket
frame that emits light uniformly in all directions.
Consider the 50 percent of this light that goes into the
forward hemisphere in the rocket frame. Also, assume
that the rocket moves very fast relative to the labora-
tory. Show that in the laboratory frame this light is
concentrated in a narrow forward cone whose axis lies
in the direction of motion of the particle. This effect is
called the *headlight effect*.

Solution: Observers A and B are at rest with
respect to observer O. They are also equidistant
from observer O, as he can verify with a meter
stick at his leisure. Therefore flashes from A and
B require equal times to arrive at O. Flashes from
A and B are observed to arrive at O at the same
time. Observer O concludes, therefore, that ob-
servers A and B emitted their flashes at the same
time: $\Delta t_{BA} = 0$.

Observer O' standing beside the rails draws an
entirely different conclusion. He reasons as fol-
lows: "The two flashes arrived when the middle
of the train (observer O) was passing me. There-
fore the two flashes must both have been emitted
before the middle of the train reached me. Before
the middle of the train reached me, observer A
was nearer to me than was observer B. Thus

light from B had farther to travel to reach me—and therefore *took a longer time* to reach me—than did light from A. But both flashes *arrived* at the same time. Therefore *observer B must have emitted his flash before observer A emitted his flash.*" ($\Delta t'_{BA} = t'_B - t'_A < 0$) In summary, observer O' beside the tracks concludes that B emitted his flash *before* A emitted his flash, while observer O riding on the train concludes that A and B emitted their flashes *at the same time*.

What is the observed difference of times between the emissions, A and B, of these flashes? In the unprimed train frame the flashes are emitted simultaneously, so $\Delta t = 0$. The separation between the emissions is $\Delta x = \Delta x_{BA} = x_B - x_A = L$ where L is the length of the train. Therefore in the primed frame (which moves to the right relative to the unprimed train frame, as is conventional in the primed-unprimed notation) the time between emissions can be found from the Lorentz transformation equation

$$\begin{aligned}\Delta t' &= -\Delta x \sinh \theta_r + \Delta t \cosh \theta_r \\ \Delta t' &= -L \sinh \theta_r = -L\beta_r/(1 - \beta_r^2)^{1/2}\end{aligned}$$

The minus sign shows that observer B (who is on the positive x' axis) emitted his flash at an *earlier*—a more negative—rocket time than observer A emitted his flash.

24. Einstein puzzler

When Einstein was a boy, he mulled over the following puzzler: A runner looks at himself in a mirror that he holds at arm's length in front of him. If he runs with nearly the speed of light, will he be able to see himself in the mirror? Analyze this question in terms of relativity.

25.* The pole and barn paradox

A worried student writes, "Relativity must be *wrong*. Consider a 20-meter pole carried so fast in the direction of its length that it appears to be only 10 meters long in the laboratory frame of reference. Therefore at some instant the pole can be entirely enclosed in a barn 10 meters long (Fig. 40). However, look at the same situation from the frame of reference of the runner. To him the *barn* appears to be contracted to half its length. How can a 20-meter pole fit into a 5-meter barn? Does not this unbelievable conclusion prove that relativity contains somewhere a fundamental logical inconsistency?"

Write a reply to the worried student explaining

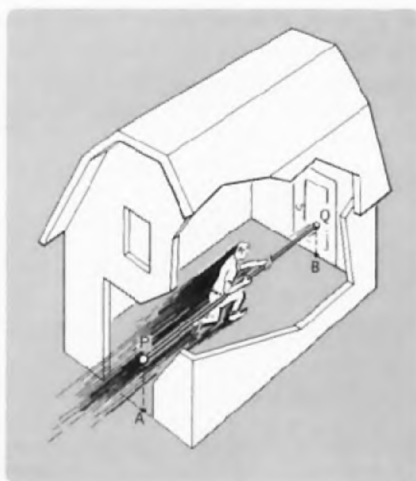


Fig. 40. Fast runner with "20-meter pole" enclosed in "10-meter barn." In the next instant he will burst through the back door, which is made of paper.

clearly and carefully how the pole and barn are treated by relativity without contradiction. (Clear up the paradox by making two carefully labeled space-time diagrams, one an xt diagram, the other an $x't'$ diagram. Take the "event" Q coinciding with A to be at the origin of both diagrams. In both plot the world lines of A, B, P, and Q. Pay attention to the *scale* of both diagrams. Label both diagrams with the times (in meters) at which Q coincides with B. Do the same for the times at which P coincides with B. Calculate these times, using the equations of the Lorentz transformation or some other method.)

26.** Space war

Two rockets of equal rest length are passing "head on" at relativistic speeds. Observer O has a gun in the

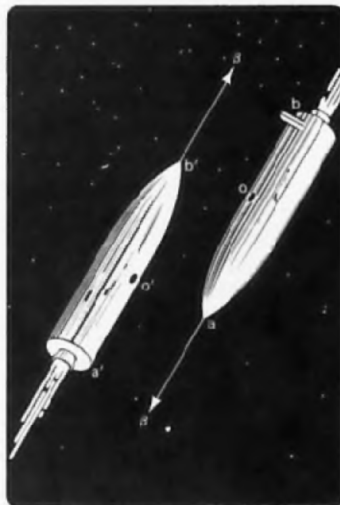


Fig. 41. Two rocket ships passing at high speed.

tail of his rocket pointing p
tion of relative motion. He f
and a' coincide:



Fig. 42. In frame of o one ex
coincides with a' to *miss* other

In the frame of O the other r
tracted. Therefore O expects
other rocket. But in the fram
O', it is the rocket ship of O t
contracted. Therefore when
observer O' sees



Fig. 43. In frame of o' one ex
coincides with a' to *hit* other sh



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Q coincides with B. Do the
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est length are passing "head
Observer O has a gun in the



tail of his rocket pointing perpendicular to the direc-
tion of relative motion. He fires the gun when points a
and a' coincide:

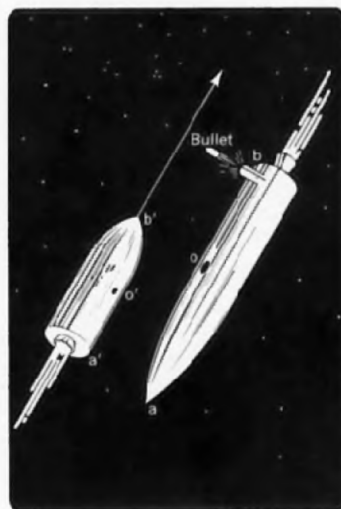


Fig. 42. In frame of o one expects a bullet fired when a
coincides with a' to miss other ship.

In the frame of O the other rocket ship is Lorentz con-
tracted. Therefore O expects his bullet to miss the
other rocket. But in the frame of the other observer,
 O' , it is the rocket ship of O that appears to be Lorentz
contracted. Therefore when points a and a' coincide,
observer O' sees

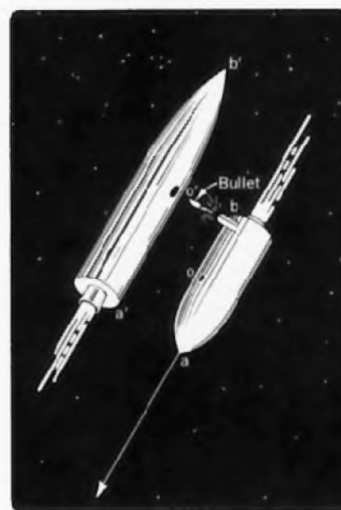


Fig. 43. In frame of o' one expects a bullet fired when a
coincides with a' to hit other ship.

Does the bullet actually hit or miss? Discuss your
answer: Pinpoint the looseness of the language used
to state the problem, also the error in one diagram.

27.* The clock paradox†

Version I; other versions in Exs. 49, 51, and 81

On their twenty-first birthday, Peter leaves his twin
Paul behind on the earth and goes off in the x direction
for seven years of *his* time (2.2×10^8 seconds or
 6.6×10^{16} meters of time) at $(24/25) = 0.96$ the speed
of light, then reverses direction and in another seven
years of his time returns at the same speed. (a) What
is Peter's age on his return? (b) Make a spacetime
diagram showing the motion of Peter. Indicate on it
the x and t coordinates of the turn-around point
and of the point of reunion. For simplicity idealize the
earth as an inertial frame, adopt this inertial frame in
the construction of the diagram, and take the origin
to be the event of departure. (c) How old is Paul at the
moment of reunion?

28.* Things that move faster than light‡

The Lorentz transformation equations have no
meaning if the relative velocity of the two frames is
greater than the velocity of light. This is taken to imply
that mass, energy, and information (messages) cannot
be moved from place to place faster than the speed of
light. Check this implication in the following examples.

(a) The scissors paradox. A very long straight rod,
which is inclined at an angle ϕ with the x axis, moves
downward with uniform speed β_y (Fig. 44). Find the
speed β_A of the point of intersection A of the lower
edge of the stick with the x axis. Can this speed be
greater than the speed of light? Can it be used to
transmit a message from the origin to someone far out
on the x axis?

(b) Suppose that the same rod is initially at rest with
the point of intersection A at the origin. The region of
the rod which is centered on the origin is struck by the
downward blow of a hammer. The point of intersec-
tion moves to the right. Can this motion of the point
of intersection be used to transmit a message faster
than the speed of light?

†For reprints of several articles on the clock paradox,
together with references to many more articles, see *Special
Relativity Theory*, Selected Reprints, published for the
American Association of Physics Teachers by the American
Institute of Physics, 335 East 45th Street, New York 17,
New York, 1963.

‡See Milton A. Rothman, "Things that go Faster than
Light," *Scientific American* 203, 142 (July, 1960).

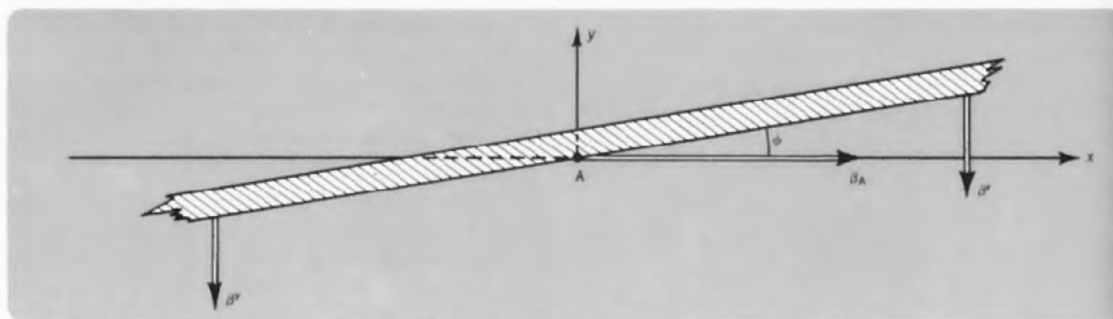


Fig. 44. Can the point of intersection A move with a speed greater than the speed of light?

(c) A very powerful searchlight is rotated rapidly in such a way that its beam sweeps out a flat plane. Observers A and B are on the plane and each the same distance from the searchlight but not near to each other. How far from the searchlight must A and B be in order that the searchlight beam will sweep from A to B faster than a light signal could travel from A to B? Before they took their positions, the two observers were given the following instructions:

To A: "When you see the searchlight beam, fire a bullet at B."

To B: "When you see the searchlight beam, duck because A has fired a bullet at you."

Under these circumstances, has not a warning gone from A to B with a speed faster than that of light?

(d) The manufacturers of some oscilloscopes claim writing speeds in excess of the speed of light. Is this possible?

D. BACKGROUND

29. Synchronization by a traveling clock—a worked example

Mr. Engelsberg does not approve of our method of synchronizing clocks by light flashes (Sect. 4). "I can synchronize my clocks in any way I choose," he says. Is he right? Mr. Engelsberg wishes to synchronize two identical clocks, named Big Ben and Little Ben, which are separated by one million miles (a little more than 1.5×10^9 meters) and which have zero relative velocity. He uses a third clock, identical in construction to the first two, that travels with constant velocity between them. As his moving clock passes Big Ben, it is set to read the same time as Big Ben. When the moving clock passes Little Ben, that outpost clock is set to read the same time as the traveling clock. "Now Big Ben and Little Ben are synchronized," says Mr.

Engelsberg. Is he right? How much out of synchronism are Big and Little Ben as measured by a lattice-work of clocks—at rest relative to them both—that has been synchronized in the conventional manner using light flashes? Evaluate this lack of synchronism when the traveling clock that Mr. Engelsberg uses moves at one hundred thousand miles per hour (4.5×10^4 meters per second). Is there any earthly reason—aside from matters of personal preference—why we all should not adopt the method of synchronization used by Mr. Engelsberg?

Solution: Start with the numerical part of the solution. A latticework of clocks at rest with respect to Big Ben and Little Ben—a latticework whose clocks are synchronized by the standard method using light flashes—can be used to make observations of the traveling clock. Relative to this latticework the traveling clock moves at $v = 4.5 \times 10^4$ meters per second, or $\beta = v/c = \frac{4.5 \times 10^4 \text{ meters per second}}{3 \times 10^8 \text{ meters per second}} = 1.5 \times 10^{-4}$ meters

of distance per meter of light-travel time. At this rate it covers the distance between Big Ben and Little Ben in a time, $\Delta t = 10^3$ meters of light-travel time. Comparison of readings of the lattice clocks with the traveling clock as it passes these in turn will show the phenomenon of *time dilation* (Ex. 10). With respect to the lattice clocks the traveling clock will run slow by a factor $(1 - \beta^2)^{1/2}$. Therefore, the time, $\Delta t'$, of travel between Big Ben and Little Ben as recorded on the *traveling clock* is

$$\begin{aligned}\Delta t' &= \Delta t(1 - \beta^2)^{1/2} \\ &= \Delta t(1 - 2.25 \times 10^{-8})^{1/2}\end{aligned}$$

Use the binomial expansion

$$(1 - \delta)^{1/2} = 1 - (\delta/2) - (1/8)\delta^2 - \dots = 1 - \delta/2 \quad (\text{for small } \delta)$$

to give an approximate

$$\Delta t' = \Delta t - (1/2)\Delta t\delta$$

or

$$\begin{aligned}(51) \quad \Delta t' - \Delta t &= \\ &= -1.12 \times 10^6 \text{ meter}\end{aligned}$$

Set Little Ben by the compare its reading w lattice. Little Ben will t lattice clocks by 0.4 mil

There is a more direc time $\Delta t'$ recorded by t way from Big Ben to straight. The lapse of t this world line is there length of the world lin events; that is, equal t passages past Big and L

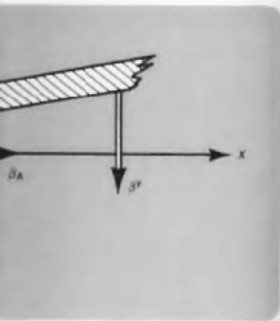
$$\Delta t' = \Delta(\text{proper time}) =$$

This calculation gives t discrepancy between lab traveling clock

$$\Delta t' - \Delta t = [(\Delta t)^2 - (\Delta x)^2/c^2]^{1/2}$$

in complete agreement

Now return to con traveling-clock method tion of clocks. Mr. En synchronization in any if he uses the traveling- nize Big Ben and Little ing difficulties: (1) Th laboratory clocks by t zation will depend on clock. Let the travelin faster than the speed g Then the discrepancy nearby lattice clocks w but about 40 millisecon by side that are syn clocks moving at differ with each other! (2) E limited to a given sp method of synchroniz path of the traveling c taken by the traveling c earlier will Little Ben r lattice clocks. (3) If th round trip from Big B



How much out of synchronism as measured by a lattice-clock relative to them both—that has been done in a conventional manner using the lack of synchronism when Mr. Engelsberg uses moves at 10 miles per hour (4.5×10^4 m/sec) for any earthly reason—aside from the lack of preference—why we all use the method of synchronization used

the numerical part of the world line of clocks at rest with respect to Little Ben—a latticework of clocks synchronized by the standard method—can be used to make the traveling clock. Relative to the traveling clock moves at 10 miles per second, or $\beta = v/c = 1.5 \times 10^{-4}$ meters per second

of light-travel time. At this distance between Big Ben and Little Ben $\Delta t = 10^{13}$ meters of light-travel time. The difference of readings of the lattice clock as it passes these clocks is the phenomenon of *time dilation*. Relative to the lattice clocks the traveling clock slows by a factor $(1 - \beta^2)^{1/2}$. The difference of readings of the lattice clock as it passes these clocks is the phenomenon of *time dilation*. The difference of readings of the lattice clock as it passes these clocks is the phenomenon of *time dilation*.

— $\beta^2)^{1/2}$
— $2.25 \times 10^{-8})^{1/2}$
ansion
— $(1/8)\delta^2 - \dots = 1 - \delta/2$
(for small δ)

to give an approximate answer

$$\Delta t' = \Delta t - (1/2) 2.25 \times 10^{-8} \Delta t$$

or

$$(51) \quad \Delta t' - \Delta t = -1.12 \times 10^{-8} \times 10^{13} \\ = -1.12 \times 10^5 \text{ meters} = -0.4 \times 10^{-3} \text{ seconds}$$

Set Little Ben by the traveling clock and then compare its reading with nearby clocks of the lattice. Little Ben will then read *earlier* than the lattice clocks by 0.4 millisecond.

There is a more direct way to find the lapse of time $\Delta t'$ recorded by the traveling clock on its way from Big Ben to Little Ben. The route is straight. The lapse of traveling-clock time along this world line is therefore equal to the proper length of the world line itself between the two events; that is, equal to the *interval* between the passages past Big and Little Ben:

$$\Delta t' = \Delta(\text{proper time}) = (\text{interval}) \\ = [(\Delta t)^2 - (\Delta x)^2]^{1/2}$$

This calculation gives the result for the time discrepancy between laboratory clocks and the traveling clock

$$\Delta t' - \Delta t = [(\Delta t)^2 - (\Delta x)^2]^{1/2} - \Delta t$$

in complete agreement with Eq. 51.

Now return to consider the validity of the traveling-clock method of defining synchronization of clocks. Mr. Engelsberg is free to define synchronization in any way he wishes. However, if he uses the traveling-clock method to synchronize Big Ben and Little Ben he will find the following difficulties: (1) The settings one gives to laboratory clocks by this method of synchronization will depend on the *speed* of the traveling clock. Let the traveling clock move ten times faster than the speed given in the example above. Then the discrepancy between Little Ben and nearby lattice clocks will be not 0.4 milliseconds but about 40 milliseconds. Two Little Bens side by side that are synchronized using traveling clocks moving at different speeds will not agree with each other! (2) Even if traveling clocks are limited to a given speed, the results of this method of synchronization will depend on the *path* of the traveling clock. The longer the path taken by the traveling clock at its fixed speed, the earlier will Little Ben read as compared to nearby lattice clocks. (3) If the traveling clock makes a *round trip* from Big Ben, it will not be synchro-

nized with Big Ben on its return! (The clock paradox, Ex. 27.) There are other inconveniences that result from Mr. Engelsberg's method of synchronization, but these are enough to show its inappropriateness for any simple description of what goes on in spacetime.

30. Time dilation and construction of clocks

In describing the phenomenon of time dilation in Ex. 10 we made no distinction between spring clocks, quartz crystal clocks, biological clocks (aging), atomic clocks, radioactive clocks, and a clock in which the ticking element is a pulse of light flashing back and forth between two mirrors. Let all these clocks be adjusted to run at the same rate when at rest in the rocket frame. When these clocks fly past standard clocks in the laboratory frame, show that the phenomenon of time dilation (Ex. 10) occurs quite independently of the inner workings of the clocks. (Discussion: How does it happen that the construction of the clocks never came into discussion before? With flashes of light flying back and forth from one clock to another for purposes of synchronization, is any clock machinery really needed? Did one ever need anything more than a first light pulse, say from an electric spark, and half-silvered mirrors at measured locations here and there (Fig. 45) to create definite time delays?)

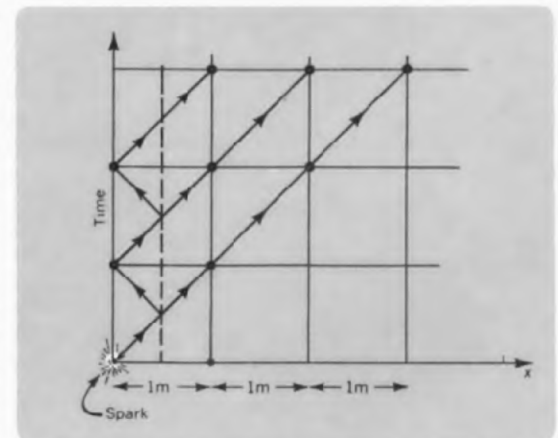


Fig. 45. Measurement of time using no clocks. Dashed line is world line of half-silvered mirror.

31. Earthbound inertial reference frames

A reference frame is inertial within some region of space and time if test particles at rest remain at rest

within some specified accuracy throughout that region of spacetime. A spaceship in free fall near the earth has been shown to be effectively an inertial frame for time periods of a few seconds. Many experiments involving fast-moving particles and light itself are observed in earthbound laboratories, which are not in free fall! The force of gravity is present in an earthbound laboratory. Nevertheless, some of these experiments require so little time that a test particle released as the experiment begins does not fall very far before the experiment is over. For the purposes of many experiments, therefore, the earthbound laboratory is an inertial frame with considerable accuracy.

(a) An elementary particle with a velocity 0.96 that of light passes through a cubical spark chamber with edges 1 meter long. How far will a separate test particle released from rest fall in the gravitational field of the earth in this time? Compare your answer

with the dimensions of an atomic nucleus (a few times 10^{-15} meters). Summarize by stating the dimensions of the spacetime region in which the laboratory or earthbound frame is idealized to be inertial and the specified accuracy. How big would the spark chamber have to be in order that the separate test particle would drop a measurable amount from rest in the time that an elementary particle of speed $0.96c$ traverses the chamber?

(b) In the Michelson-Morley experiment (Ex. 33) a beam of light is reflected back and forth between pairs of mirrors about 2 meters apart so that it travels a total distance of 22 meters. How far will a test particle fall from rest in the gravitational field of the earth during the time that a particular photon traverses the Michelson-Morley equipment? To what accuracy is the earthbound frame inertial in the spacetime region in which the Michelson-Morley experiment is carried out?



Fig. 46. Ball bearings released from rest will move closer together as they descend (to scale.)

32.* Size of an inertial frame

How large can a given region of space be ($\Delta x = \Delta y = \Delta z = L$, meters), how long can it be studied (Δt , meters!), and how close can it be to a center of gravitational attraction, before a detectable discrepancy, ϵ , from an ideal inertial reference system shows up?

(a) *One kind of discrepancy: relative acceleration perpendicular to the line of attraction.*

(1) Special case. Two ball bearings are released from rest from a common height of 250 meters

above the earth and 25 meters apart (Fig. 46). Show that they will move closer together by a distance of about 10^{-3} meter before striking the earth. (Analyze by the method of similar triangles or by some other method. This is the example treated on page 9 of the text.) The time to fall 250 meters at an acceleration of 9.8 meters per second per second is about 7 seconds or 21×10^8 meters of light-travel time. In summary, a falling railway coach can be treated as an inertial reference system under these conditions:

Conditions that are adequate to guarantee that discrepancy from ideal inertial frame cannot be detected

ϵ (smallest discrepancy given instruments can detect)	r (distance from center of earth)	Δx (horizontal spread)	Δy and Δz (spread of region in other two directions)	Δt (time of observation)
$\epsilon \geq 1 \times 10^{-3}$ meter	$r \geq r_e = 6.4 \times 10^6$ meters	$\Delta x = L \leq 25$ meters	Assumed zero in analysis; therefore have to be assumed zero here in default of further analysis (part c)	$\Delta t \leq 21 \times 10^8$ meters (7 seconds)

(2) More general case. Two ball bearings are placed Δx relative to each other at the same distance r from the center of attraction, and are studied for a time Δt . Let a be the common acceleration towards the center of attraction. The acceleration is a second per second, or a meters per second per second. Show that the acceleration discrepancy is Δa^2 (meters per meter of time squared).

$$(52) \quad (\Delta a^2)^* = \frac{a^2 \Delta x^2}{c^2 \Delta t^2}$$

(Assume that the relative acceleration is small that the sine and the tangent can all be identified with the angle.)

(b) *Another kind of discrepancy: relative acceleration parallel to the line of attraction.*
(1) General case. Test particle A is at distance r from the center of attraction. Therefore B is further from the center of attraction than A. The acceleration is a second per second, or, as seen by an observer at the center of attraction, a meters per second per second. Show that this relative acceleration is Δa^2 (meters per meter of time squared).

$$(53) \quad (\Delta a^2)^* = \frac{a^2 \Delta x^2}{c^2 \Delta t^2}$$

atomic nucleus (a few times by stating the dimensions of which the laboratory or earth- to be inertial and the specified the spark chamber have to te test particle would drop a rest in the time that an ele- 0.96 c traverses the chamber? Morley experiment (Ex. 33) a back and forth between pairs s apart so that it travels a . How far will a test particle vitational field of the earth ticular photon traverses the ment? To what accuracy is tial in the spacetime region Morley experiment is carried

1 25 meters apart (Fig. 46). move closer together by a 10^{-3} meter before striking the the method of similar tri- other method. This is the page 9 of the text.) The ers at an acceleration of 9.8 er second is about 7 seconds of light-travel time. In sum- way coach can be treated as e system under these con-

that discrepancy detected

Δz region two (s)	Δt (time of observation)
ero in efore sumed default analysis (s)	$\Delta t \leq$ 21×10^8 meters (7 seconds)

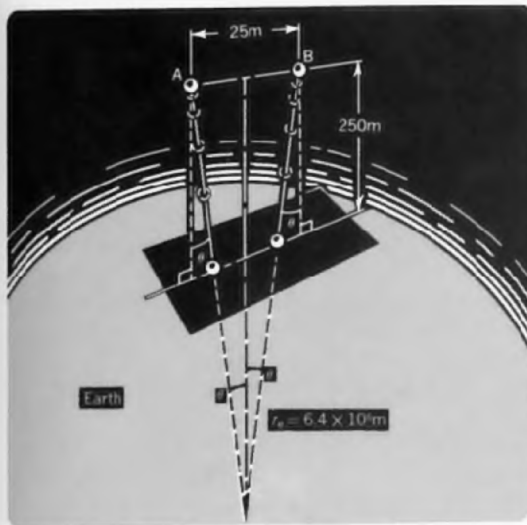


Fig. 46. Ball bearings released side by side near the earth move closer together as they descend. (Figure is not drawn to scale.)

(2) More general case. Test particle B is displaced Δx relative to test particle A. They have the same distance r from the center of attraction, and are studied for a time Δt . Denote by a the common acceleration of the two particles towards the center of attraction in meters per second per second, or by $a^* = a/c^2$ the value of the same acceleration measured in meters of distance per meter of time per meter of time. Show that the acceleration of particle B relative to A $(\Delta a^*)^*$ (meters of distance per meter of time per meter of time) is given by the formula

$$(52) \quad (\Delta a^*)^* = -(\Delta x/r)a^*$$

(Assume that the relevant angles are so small that the sine and the tangent and the angle itself can all be identified with one another.)

(b) Another kind of discrepancy: relative acceleration parallel to the line of attraction.

(1) General case. Test particle B is displaced by the amount Δz relative to A and parallel to r . Therefore B is further away from the center of attraction than A and experiences a smaller attraction. Consequently B is left behind A—or, as seen by an observer on A, B has a relative acceleration in the positive z direction. Show that this relative acceleration (in meters of distance per meter of time per meter of time) is

$$(53) \quad (\Delta a^*)^* = +(2 \Delta z/r)a^*$$

(Hint: Use the fact that a^* falls off according to Newton's inverse square law of gravitation; thus, $a^* = \text{constant}/r^2$. Evaluate at r and at $r + \Delta z$ and take the difference. Take advantage of the fact that Δz is very small (a few meters) compared to r (thousands of kilometers) to simplify the result!)

(2) Special case. (Page 9 of text.) One test particle is 250 meters above the surface of the earth, the other 275 meters. How much will the 25 meter separation between the particles be increased in the approximately 7 seconds it takes for the first particle to hit the ground? (Hint: By what factor do the expressions for Δa^* in part b,1, and Δa^* in part a,2, differ from each other?) Use your result to complete—or, if you wish, to revise—the table in part a,1.

(c) Case in which the region of experimentation is far from the center of the earth.

The space corporation increases the scope of the experiments with test particles and light rays. The research group finds that the region used for previous experiments is not large enough for the new program and 7 seconds is not long enough. Management agrees to its recommendation for a space $\Delta x = 200$ meters, $\Delta y = 200$ meters, $\Delta z = 100$ meters and a time of 100 seconds, with the same tolerance as before, $\epsilon = 1 \times 10^{-3}$ meters = 1 millimeter. To how many earth radii from the center of the earth must the equipment be boosted by rockets to make the departures from ideality less than the tolerable upper limit? (Some possible questions to ask along the way: How does a^* vary with the distance r from the center of the earth? How do $(\Delta a^*)^*$ and $(\Delta a^*)^*$ vary with r ? How do Δx and Δz depend upon $(\Delta a^*)^*$ and $(\Delta a^*)^*$ and the time Δt ?)

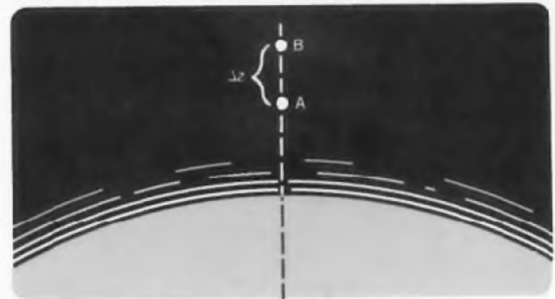


Fig. 47. Ball bearings released one above the other near the earth move farther apart as they descend. (Figure is not drawn to scale.)

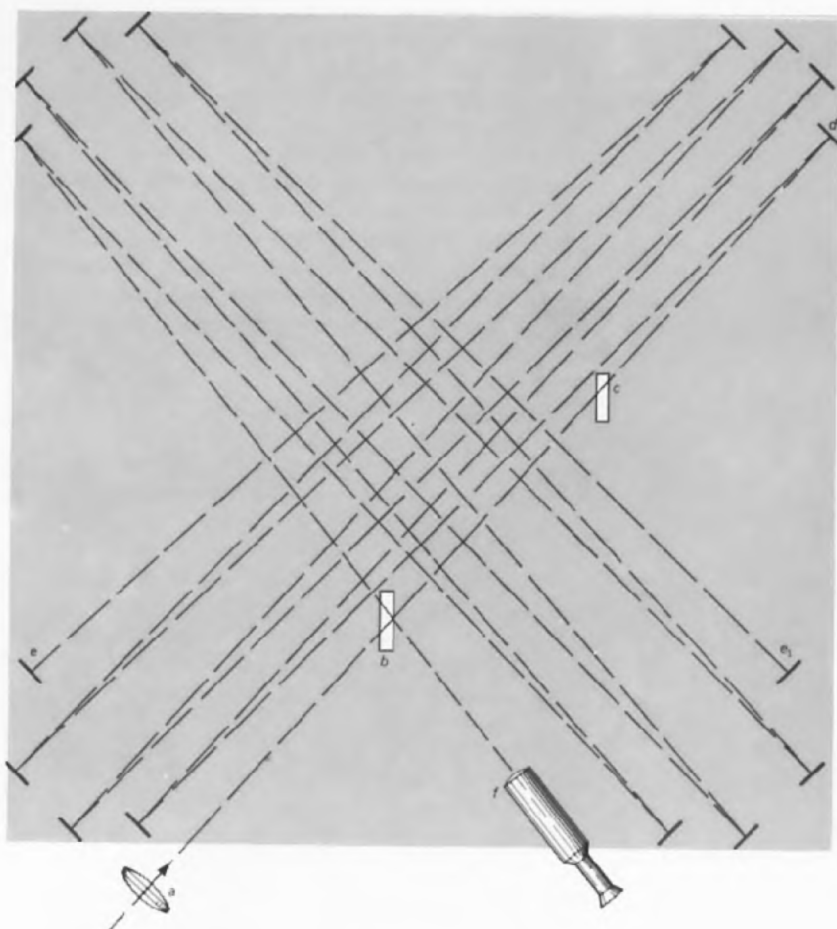


Fig. 48. Michelson-Morley interferometer mounted on a rotating marble slab.

33.* The Michelson-Morley experiment†

(a) An airplane moves with air speed c (not the speed of light) from point A to point B on the earth. A stiff wind of speed v is blowing from B toward A. Show that the time for a round trip from A to B and back to A under these circumstances is greater by a factor $1/(1 - v^2/c^2)$ than the corresponding round-trip time in still air. Paradox: The wind helps on one leg of the flight as well as hinders on the other: Why, therefore, is the round-trip time not the same in the presence of wind as in still air? Give a simple physical

reason for this difference. What happens when the wind speed is nearly equal to the speed of the airplane?

(b) The same airplane now makes a round trip between A and C. The distance between A and C is the same as the distance from A to B, but the line from A to C is perpendicular to the line from A to B, so that in moving between A and C the plane flies *across* the wind. Show that the round-trip time between A and C under these circumstances is greater by a factor $1/(1 - v^2/c^2)^{1/2}$ than the corresponding round-trip time in still air.

(c) Two airplanes with the same air speed c start from A at the same time. One travels from A to B and back to A, flying first against and then with the wind (wind speed v). The other travels from A to C and back to A, flying across the wind. Which one will arrive

home first, and what will be the difference in arrival times? Using the binomial expansion, $v \ll c$, then an approximate

$$\text{difference is } \Delta t = \frac{L}{2c} \left(\frac{v^2}{c^2} \right),$$

distance between A and B (and C).

(d) The South Pole Air Station in Antarctica is a good place for research huts on a circle centered on the air station. Supply planes start simultaneously from the station and fly radially in all directions. Each plane drops supplies at a search hut and flies directly back to the station. A stopwatch stands on the station. He notices that the planes arrive at the same time. This discrepancy is caused because he knows from careful measurements that (1) the distance from the air station to each hut is the same, (2) every plane flies with the same speed as every other plane—and (3) every plane travels in a straight line on the ground from station to hut and back. He finally decides that the discrepancy is not due to the stopwatches at the high altitude at which they are used. With the stopwatch he measures the time for the first plane to the return of the last plane. What is the wind speed of the planes fly? What can the Earth's rotation tell you about the direction of this wind?

(e) In their famous experiment, Michelson and Morley attempted to detect the motion of the earth through the ether with respect to which light was supposed to travel at speed c . They compared the round-trip time for light to travel equal distances parallel and perpendicular to the direction of motion of the earth. The light was reflected the light back and forth between two parallel mirrors. (This would be done by each airplane made repeated trips between the mirrors means they were able to use a distance of 22 meters for each path. If the earth moves with respect to the sun, and if the speed of light is 300,000 meters per second in its path, the approximate difference in time for the light flashes that are emitted from the station and travel along the two perpendicular paths. With the instruments of today, the difference in time is so small that the ether-drift hypothesis was never measured directly, and the following experiment is instead.

†A. A. Michelson and E. W. Morley, *American Journal of Science*, **34**, 333 (1887). The logical position of the experiment in the theory of relativity is outlined by H. P. Robertson, in *Reviews of Modern Physics*, **21**, 378 (1949).