tromagnetic radiation, gravitational radiation, and neutrinos—and of these only the first and the last have so far been observed experimentally.†

The relation $p = E$ is fulfilled with 100 percent accuracy only by a radiation of zero rest mass. However, it is approximated with arbitrarily high precision by a particle that has an energy sufficiently great in comparison with its rest mass. Therefore in this extreme relativistic limit a particle of rest mass $m$ behaves—so far as concerns the laws of conservation of momentum and energy—in practically the same way as a photon.

## 13. The Equivalence of Energy and Rest Mass

*Rest mass of final system increases in an inelastic encounter*

The total of the momenta of all particles is conserved in a collision; the total of their energies $E$ (rest energy plus kinetic energy) is also conserved. That has been the guiding principle in our analysis of collisions. But is it correct to accept this principle when we turn from considering elastic encounters to look at an *inelastic* collision? A ball of putty is thrown with high speed against another ball of putty lying at rest on a skating rink. They stick together as they hurtle across the ice. We are quite willing to believe that the law of conservation of momentum applies to this collision; but is it reasonable to expect the law of conservation of energy to be useful in analyzing this collision? Some of the energy of impact has been converted into heat. Another portion of the original energy shows up in the energy of rotation as the two joined balls whirl like a dumbbell about their center-of-mass. How can any treatment give adequate recognition to these complications when it limits its description of the final system to the two quantities $E$ and $p$, connected by the elementary formula $E^2 - p^2 = m^2$? Answer: the rest mass $m$ of the final system is greater than the sum of the rest masses of the original colliding objects. This is a new feature of spacetime physics, never known or even imagined in Newtonian mechanics. The increase in rest mass measures precisely the energy which has gone into heat and whirling and any other forms of internal excitation of the final system. Unless one recognizes the changes in rest mass that take place in many encounters, he will be led into apparent violations of the law of conservation of energy or the law of conservation of momentum or both.

How does one evaluate the alteration of the rest mass? In the example of the two balls of putty he applies (1) the law of conservation of energy

$$E_{\text{final}} = E_{\text{initial}} = E_1 + m_2$$

(2) the law of conservation of momentum

$$p_{\text{final}} = p_{\text{initial}} = p_1 + p_2 = p_1$$

and (3) the equation

$$m_{\text{final}}^2 = E_{\text{final}}^2 - p_{\text{final}}^2$$

†On the detection of free neutrinos see C. L. Cowan, Jr., F. Reines, F. B. Harrison, H. W. Kruse, and A. D. McGuire, Science, **124**, 103 (1956).

For attempts presently in progress to detect gravitational radiation from outer space, see J. Weber, "Gravitational Waves," in *Gravitation and Relativity*, edited by H.-Y. Chiu and W. F. Hoffmann, (W. A. Benjamin, New York, 1964).

He obtains the result

$$
\begin{aligned}
m_{\text{final}}^2 &= (E_1 + m_2)^2 - p_1^2 \\
&= E_1^2 + 2E_1 m_2 + m_2^2 - p_1^2 \\
&= (E_1^2 - p_1^2) + 2E_1 m_2 + m_2^2 \\
&= m_1^2 + 2(m_1 + T_1)\, m_2 + m_2^2 \\
&= (m_1 + m_2)^2 + 2T_1 m_2 \qquad (92)
\end{aligned}
$$

*Conservation laws valid whether collision is elastic, inelastic, or superelastic*

Evidently the rest mass of the agglomerated system exceeds the sum of the rest masses of the original objects 1 and 2. Moreover, the amount of the excess is greater when the kinetic energy of impact, $T$, is large. We conclude from this example that *the laws of conservation of momentum and energy are as valid*—and as useful—*for inelastic encounters as for elastic processes.*

How does this unexpected bonus from the conservation laws come about? And what does it tell us about the equivalence of energy and rest mass? These questions require a closer look.

"Energy is conserved in every frame if momentum is conserved in both laboratory and rocket frames." In the proof of this theorem in Eqs. 79 and 80 it made no difference whether a single object emerged from the collision or a thousand fragments came off, or whether the two colliding particles met in an elastic encounter. There are many reactions in physics that change the number of particles. One of the most dramatic is the creation of a pair of electrons, one positive, the other negative, out of empty space during the collision of two energy-bearing agencies; for example, in the collision of two electrons

$$e^-(\text{fast}) + e^-(\text{at rest}) = e^- + e^- + e^- + e^+$$

Such a process is called inelastic because kinetic energy is converted into rest mass. There are also superelastic processes, in which part of the rest mass of an object (stored internal energy) is converted into kinetic energy

$$\begin{pmatrix}\text{slow}\\ \text{electron}\end{pmatrix} + \begin{pmatrix}\text{atom containing}\\ \text{internal excitation energy}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{deexcited}\\ \text{atom}\end{pmatrix} + \begin{pmatrix}\text{fast}\\ \text{electron}\end{pmatrix}$$

Finally, there are decay processes, in which one particle breaks up into two products of lesser combined rest mass

$$K^+ \longrightarrow \pi^+ + \pi^0$$

(positive $K$-meson [mass of 967 electron masses] decaying in $10^{-8}$ second to positive pi-meson [mass of 273 electron masses] plus neutral pi-meson [mass of 264 electron masses]).

All complications that originate, in these or other ways, from changes in the number of particles in no way affect the applicability of the laws of conservation of momentum and energy. Happily the reactants and the reaction products—and their energies and momenta—can be defined and discussed whether the reaction they have just been through or are just about to go through is elastic or inelastic. Each particle carries its momentum-energy 4-vector at all times: It does not know whether it is going to go through an inelastic or an elastic encounter. It needs to have all the bookkeeping machinery for a possible *elastic* collision. Thus, whether the collision is destined to be elastic or inelastic, one knows the momentum and energy of each particle before the collision. Therefore one also knows the *total* momentum and energy of the entire system before the collision. One likewise knows the *total* momentum and energy after the collision. Therefore one can speak of the *change*—if any—in the total energy and momentum in the collision. This change is zero in an elastic encounter. The change of energy is also zero in inelastic encounters, provided that the change of total *momentum* is zero in both laboratory and rocket frames: of this we are assured by earlier reasoning (Eqs. 79, 80). No convincing argument has ever been given for doubting that momentum—and therefore energy as well—is conserved in inelastic collisions.

*Innumerable observational tests of conservation laws*

What about observational evidence on conservation of momentum and energy in inelastic events? Momentum and energy have been defined in such a way that these quantities are conserved in the simplest of elastic encounters. Therefore it is too late to change the definitions to fit observations on a wider range of collision processes. Either the change in momentum and energy is zero when measured by every experiment—in which case the conservation of momentum and energy is a principle of wide-reaching significance; or the change in momentum and energy is *not* zero—in which case the experimental result would be a revolutionary upset of the principles of relativity. Experiment makes the test—and observation confirms that this change is zero. This test is repeated daily and hourly as high-energy collisions are recorded in laboratories all over the world.

For experiments that test the conservation laws and experience in interpreting them, see Exs. 90 to 100.

**Table 11.** The number of yearly tests of Euclidean geometry and of Lorentz geometry.

| *How well tested is Euclidean geometry?* | *How well tested is relativity?* |
|---|---|
| 42,000 surveyors (1963 Statistical Abstracts of the United States), each making 20 surveys per year in which he identifies the $n$ vertices of a polygonal boundary, measures the interior angle at each vertex, adds, and compares the sum with the value $(n - 2)\ 180°$ predicted by Euclidean geometry. | 50 particle accelerators (estimated) that produce particles of energy greater than 100 MeV, each operating 100 days per year, each recording 200 collisions per day of operation in which departures from the relativistic conservation laws would be apparent. |
| *Result:* 840,000 tests per year, each with a sensitivity of one part in $10^4$ or better. | *Result:* 1,000,000 tests per year, each with a sensitivity of one part in $10^4$ or better. |

The energy set free in the burning of coal, the combustion of gas, and the explosion of dynamite appears large on an everyday level of experience. However, when one translates these numbers into mass equivalent, he finds that less than one part in $10^9$ of the rest mass has been converted into energy (see, for example, Ex. 63)—a mass change too small for detection by existing equipment. Therefore one is driven to the worlds of particle physics and nuclear physics when he looks for places to make careful tests of the conservation laws.

In particle physics many of the objects under study live for a very short time. It is not easy to determine precisely the mass of such a short-lived particle by using a conventional mass spectrometer. The mass is found instead by applying the laws of conservation of momentum and energy to a collision or transformation process in which one or more of the particles have known masses. One can still check the conservation laws in this way, for a given particle is often formed by several different reactions. However, for a direct check of energies of transmutation against the energies expected from changes in rest masses, it is better to turn to the world of nuclear physics. There mass values have been determined directly and with high precision both for stable nuclei and for some unstable nuclei. The conditions for an accurate comparison of energy release with mass changes are most favorable for the lighter nuclei. There the change of mass in the typical nuclear reaction constitutes a higher fraction of the total mass—and is therefore subject to more precise determination—than in heavier nuclei. For this reason examine the reaction of two of the lightest of atomic nuclei, a reaction which, moreover, is of great importance in this nuclear age:

*Conditions for a precise test of conservation laws especially favorable in nuclear physics*

$$\begin{pmatrix}\text{fast}\\ \text{deuteron}\end{pmatrix} + \begin{pmatrix}\text{deuteron}\\ \text{at rest}\end{pmatrix} \begin{array}{l} \nearrow \begin{pmatrix}\text{very energetic}\\ \text{proton}\end{pmatrix} + \begin{pmatrix}\text{energetic}\\ \text{triton}\end{pmatrix} \\ \\ \searrow \begin{pmatrix}\text{very energetic}\\ \text{neutron}\end{pmatrix} + \begin{pmatrix}\text{energetic}\\ \text{helium 3}\end{pmatrix} \end{array}$$

or

$$\mathrm{H}^2(\text{fast}) + \mathrm{H}^2 \begin{array}{l} \nearrow \mathrm{H}^1 + \mathrm{H}^3 \\ \searrow \mathrm{n} + \mathrm{He}^3 \end{array} \tag{93}$$

The alternative reactions of Eq. 93 occur with comparable frequency in a hydrogen bomb, or "fusion" weapon. They provide a large part of the energy release characteristic of a device fueled by deuterium ("heavy hydrogen," $\mathrm{H}^2$). The kinetic energy of the products of such a thermonuclear reaction is hundreds of times greater than the kinetic energy of the deuterons entering the reaction.

*Triton mass from conservation law checks triton mass as measured by spectrometer*

The triton-producing reaction—the first of the two alternative transformations in Eq. 93—lends itself to the most precise single test of the conservation laws that we have been able to find in all of physics. Making this test possible are independent, and careful, mass spectrometer determinations of the rest masses of all particles taking part in this reaction (deuteron, proton, triton).

An equally accurate and independent determination of the rest mass of the neutron is not available. Hence we exclude from attention the second, or neutron-producing reaction in Eq. 93. It is not suitable for a highly accurate test of the equivalence of mass and energy. The neutron is an unstable particle (mean life about 17 minutes). More important, it is unresponsive (electrically neutral!) to the electric and magnetic fields in a mass spectrometer. This unresponsiveness stands in the way of any precise independent determination of the mass of the neutron.

Imagine that we were to concern ourselves here with the neutron rather than the triton. What could we hope to discover, since we lack any independently determined *precise* value of the neutron mass? We could give up trying to *check* the conservation laws. Instead, we could *apply* the conservation laws to deduce a value for the neutron mass good to about 1 part in $10^5$. Why can we be assured that the conservation laws—as applied to the second reaction—provide a means so reliable for determining the mass of the neutron? Because, applied to the first reaction, the conservation laws give a value for the triton mass that checks the mass spectrometer value to better than 1 part in $10^5$ (ANALYSIS OF THE REACTION $H^2 + H^2 \longrightarrow H^1 + H^3$, pages 126 and 127). This one check of maximal present day precision, together with many other checks of slightly lower accuracy elsewhere in physics, argues powerfully for the soundness of the conservation principle.

A remark has to be made about the units in the calculation on pages 126 and 127. In principle it would be natural to express all energies and momenta in kilograms, as was done in earlier calculations of this chapter. To do this here, however, it would be necessary to translate the numbers given by mass spectroscopy from "unified atomic mass units" ($u$; scale changed in 1961 from $O^{16} = 16.000 \cdots$ to $C^{12} = 12.000 \cdots$) to kilograms, and also to translate the kinetic energy values measured by the nuclear physicist from electron volts to kilograms. It is more convenient to express all energies in $u$, thus eliminating the need for one of the conversion calculations ($u$ to kilograms). Moreover, all formulas in the text apply as well to one set of units of mass-energy as to another set, provided only that a given set of units is used consistently throughout. How then is one to translate from electron volts to $u$? Happily the translation can be made without knowing the mass in kilograms associated with one mass unit—or equivalently, without knowing how many atoms there are in a gram atom (Avogadro's number, $N$, which has a value of $(6.02252 \pm 0.00028) \times 10^{23}$). The present-day uncertainty in this number, 5 parts in $10^5$, would affect all our results if we chose to translate into kilograms. The conversion factor from electron volts to unified atomic mass units is derived on page 128.

# ANALYSIS† OF THE REACTION $H^2$ (FAST) $+ H^2 \rightarrow H^1 + H^3$

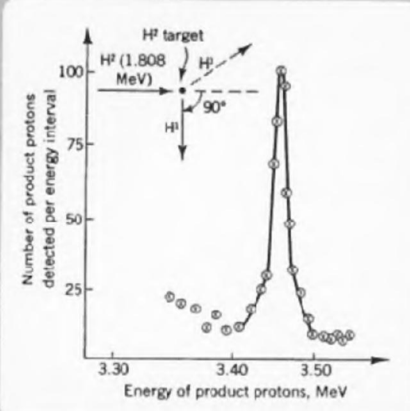


**Fig. 92.** Experimental evidence that protons ($H^1$) emerging from the reaction

$$H^2(1.808 \text{ MeV}) + H^2(\text{at rest}) \longrightarrow H^1(\text{very fast}) + H^3(\text{fast})$$

at 90 degrees from the direction of the incident deuteron ($H^2$) have an energy of 3.467 MeV. (The value 3.467 MeV was obtained by combining the results shown here with results of several similar determinations.) The number of protons emerging with energies in the range $E - 0.1$ MeV to $E + 0.1$ MeV is plotted as a function of $E$. The spread in energy arises from the finite thickness of the target, the finite width of slits used to define the beam, inhomogeneities in the magnetic fields used to define energies, etc. The experimental curve in this figure can be found in the article by D. M. Van Patter and W. W. Buechner, Physical Review, **87**, 51, (1952).

*Laws of conservation of energy and momentum:*

(94) $\quad E_2 + m_2 = \bar{E}_1 + \bar{E}_3 \quad$ energy conservation

(95) $\quad p_2^x + 0 = 0 + \bar{p}_3^x \quad$ $x$-momentum conservation

(96) $\quad 0 + 0 = \bar{p}_1^y + \bar{p}_3^y \quad$ $y$-momentum conservation

(97) $\quad 0 + 0 = 0 + \bar{p}_3^z \quad$ $z$-momentum conservation

Here each subscript indicates the mass number of the isotope and a bar over a symbol means "after the reaction."

Each of the four equations (94–97) can be regarded as giving a separate piece of information about the triton—either about its energy, or about one component of its momentum. But we are not interested in all this information. We want to know one simple property of the triton, different from any of these four quantities—its rest mass. Happily, this rest mass is given by the length of the 4-vector of energy and momentum

(98) $$m_3^2 = \bar{E}_3^2 - (\bar{p}_3^x)^2 - (\bar{p}_3^y)^2 - (\bar{p}_3^z)^2$$

Insert into this expression the values of the components read from Eqs. 94–97 and obtain

$$\begin{aligned}
m_3^2 &= (E_2 + m_2 - \bar{E}_1)^2 - (p_2^x + 0 - 0)^2 - (0 + 0 - \bar{p}_1^y)^2 - (0 + 0 - 0)^2 \\
&= \underbrace{\bar{E}_1^2 - 0 - (\bar{p}_1^y)^2 - 0}_{} + \underbrace{E_2^2 - (p_2^x)^2 - 0 - 0}_{} + m_2^2 - 2m_2\bar{E}_1 + 2m_2E_2 - 2E_2\bar{E}_1 \\
&= \qquad m_1^2 \qquad + \qquad m_2^2 \qquad + m_2^2 - 2m_2\bar{E}_1 + 2m_2E_2 - 2E_2\bar{E}_1 \\
&= m_1^2 + 2(m_2 + E_2)(m_2 - \bar{E}_1)
\end{aligned}$$

(99) $$m_3^2 = m_1^2 + 2(m_2 + m_2 + T_2)(m_2 - m_1 - \bar{T}_1)$$

Here we have used equations of the form $E = m + T$, which relate kinetic energy to total energy.

†Results of the experiment presented in this calculation are published in an article by E. N. Strait, D. M. Van Patter, W. W. Buechner, and A. Sperduto, in the Physical Review, **81**, 747 (1951). The authors express their appreciation to W. W. Buechner and A. Sperduto for additional data and for discussions regarding their proper interpretation.

The values of all quantities on the right side of Eq. 99 are known. This equation thus permits a prediction of the mass $m_3$ of the triton. Mass values for the right side of Eq. 99 are obtained from mass spectrometer experiments (expressed in "unified atomic mass units"—symbol $u$—based on the isotope $C^{12} = 12.0000\ldots u$. as the standard of mass).†

(100) $$m_2 = 2.0141019 \pm 0.0000003\ u$$

(101) $$m_1 = 1.0078252 \pm 0.0000003\ u$$

Kinetic energies were measured in the nuclear reaction experiment (Fig. 92).

(102) $$(\text{kinetic energy of incident deuteron}) = T_2$$

$$= (1.808 \pm 0.002\ \text{MeV})(1.073562 \times 10^{-3}\ u/\text{MeV}) = 0.001941 \pm 0.000002\ u$$

For derivation of the conversion factor from MeV to unified atomic mass units, see page 128.

(103) $$(\text{kinetic energy of product proton}) = \overline{T}_1$$

$$= (3.467 \pm 0.0035\ \text{MeV})(1.073562\ u/\text{MeV}) = 0.003722 \pm 0.000004\ u$$

Substitute these values into Eq. 99, keeping only the six decimal digits that correspond to the accuracy of the kinetic energy measurements. The two terms on the right of this equation become

$$\begin{aligned} m_1^2 &= 1.015712\ u^2 \\ 2(2m_2 + T_2)(m_2 - m_1 - \overline{T}_1) &= \underline{8.080881 \pm 0.00003\ u^2} \\ \text{sum of these terms} = m_3^2 &= 9.096593 \pm 0.00003\ u^2 \end{aligned}$$

The square root of this number is the mass of the triton predicted on the basis of this nuclear reaction.

(104) $$m_3 = 3.016056 \pm 0.000015\ u$$

Compare this mass with the mass of the triton measured by mass spectrometer.†

(105) $$m_3 = 3.0160494 \pm 0.0000007\ u$$

The two values differ by about 2 parts in $10^6$, a difference less than the uncertainty in the result of the nuclear reaction experiment. So accurate a prediction of the triton mass on the basis of the relativistic conservation laws is strong evidence for the validity of these conservation laws.

---

†Mass spectrometer mass values quoted here are derived from F. Everling, L. A. König, J. H. E. Mattauch, and A. H. Wapstra, Nuclear Physics, **25**, 177 (1961). Mass values could instead have been taken from standard tables of nuclidic masses. However, mass values in standard tables constitute a "best compromise" between various kinds of data, including not only mass spectrometer results but also the results of nuclear reaction experiments such as the one under discussion here. In constructing standard tables, data from nuclear reaction experiments are interpreted using the conservation laws of relativity. Therefore mass values from standard tables cannot be used in an independent check of these conservation laws, as we would like to do here. For this reason we limit present attention to mass spectrometer mass values. After this and other similar verifications of the conservations laws, we are willing to place reliance on standard tables, which use these laws to derive the best possible mass values from available data of all kinds. Such a standard mass table can be found in L. A. König, J. H. E. Mattauch, and A. H. Wapstra, Nuclear Physics, **31**, 18 (1962).

*Conversion factor from electron volts to unified atomic mass units*

One electron volt is the amount of energy required to move one particle with an elementary charge $e$ through a voltage difference of one volt

↓ multiply by $N$

$N$ electron volts is the energy to move $N$ elementary charges through one volt

↓

or the energy to move a mole of electricity through one volt

↓

or the energy to move one Faraday electrochemical equivalent ($96487.0 \pm 1.6$ coulomb) through one volt; that is, $96487.0 \pm 1.6$ joules

↓

This is $N$ electron volts.

One $u$? Take 12 grams of $C^{12}$; divide by $N$; get the mass of one atom of $C^{12}$; it is 12 $u$ by definition. Therefore $1\ u = (1/N)$ gram

↓ multiply by $N$

$N$ atomic mass units, expressed as a mass, is equivalent to one gram, or 0.001 kilogram

↓

or equivalent in energy units to $mc^2 = 0.001$ kilogram $\times$ [$(2.997925 \pm 0.000003) \times 10^8$ meters/second]$^2$

↓

or equivalent to $(8.987554 \pm 0.000009) \times 10^{13}$ joules

↓

This is $N$ unified atomic mass units.

↘ ↙

Therefore 1 electron volt is equivalent to

$$\frac{96487.0 \pm 1.6}{(8.987554 \pm 0.000009) \times 10^{13}}$$

or $(1.073562 \pm 0.000017) \times 10^{-9}\ u$
(uncertainty 1.6 parts in $10^5$)

↓

or 1 $u$ is equivalent to
$(0.931478 \pm 0.000015) \times 10^3$ MeV

This conversion factor is used in the calculation on pages 126 and 127 to translate the observed kinetic energies of the deuteron and triton from electron volts to $u$, as needed for the final check of the conservation law.

*Not all kinetic energies measured: simple energy comparison not possible*

In that calculation one sees the mass value of a triton as obtained from the conservation laws, compared with and checked against the mass spectrometer determination. The sample verification of spacetime physics is impressive. One cannot doubt that rest mass energy is converted into kinetic energy. But one can still be puzzled at just how this simple principle gets translated into an equation so complicated as that employed in the checking process (Eq. 99). Why did we not take the spectrometer masses of the reactants, similar mass values for the products, and compare them with the balance of kinetic energies in the transmutation process? What could be simpler!

| | | | |
|---|---|---|---|
| *Reactants:* | $H^2$ | 2.0141019 *u* | |
| | $H^2$ | 2.0141019 *u* | |
| | | Sum: | 4.0282038 *u* |
| *Products:* | $H^1$ | 1.0078252 *u* | |
| | $H^3$ | 3.0160494 *u* | |
| | | Sum: | 4.0238746 *u* |
| | | Difference: | 0.0043292 *u* |
| | | equivalent to: | 4.0322546 MeV |

The difficulty arises only in the next step: evaluating from observation the net release of kinetic energy. The kinetic energy of the deuteron that moves before the reaction is known to be 1.808 Mev, and the kinetic energy of the proton after the reaction, 3.467 MeV. However, it was not convenient to measure at the same time the kinetic energy of the emerging triton, and this energy was not measured. If that kinetic energy was not known, then the net release of kinetic energy in the reaction was not directly measured. So how was it possible to test energy release versus change in rest mass? Or did our comparison really amount to any such direct confrontation of two energies? No!

One obtains a false impression of what is going on if he thinks of it as concerned exclusively with energy. An equally misleading way to do surveying is easily described. The plot of ground is a polygon of an odd shape laid out on a surface which is not flat. The length of a straight boundary AB is desired. The surveyor has measured only the difference in north-south coordinates of A and B. He will obviously be in difficulty if this is the limit of his information! Similarly, there is no hope of finding the mass of the triton from the deuteron-deuteron reaction data used above if one considers energy alone. He must also take into account the balance of momentum.

*Finding triton mass is analogous to finding length of a sloping side of polygon*

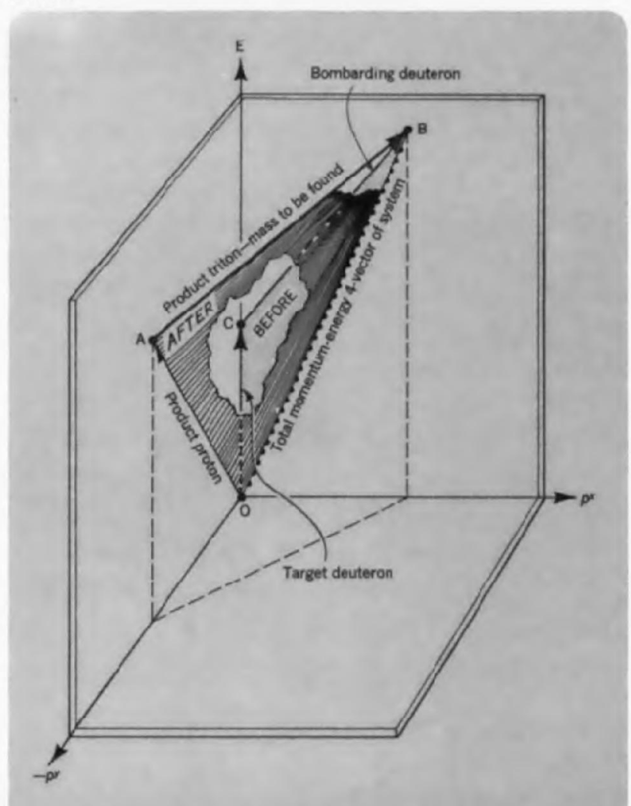


**Fig. 93.** Finding the triton mass—by using the conservation laws—treated as a problem in geometry. Note: Points O, B, C lie in the plane of the paper; point A lies above the plane of the paper ($y$ component of momentum).

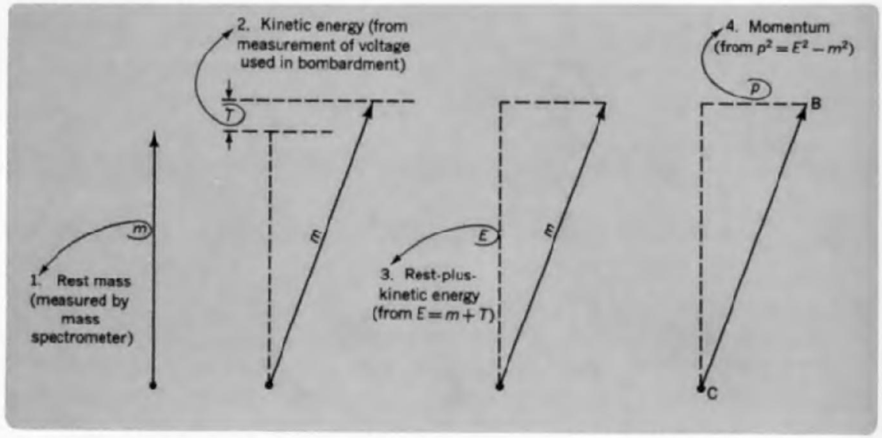


**Fig. 94.** Finding from experiment the energy and momentum components of the momentum-energy 4-vector of the bombarding deuteron in the $H^2 + H^2 \longrightarrow H^1 + H^3$ experiment. (The labels B, C for the two ends of the vector refer back to the labeling in Fig. 93.)

*One determines the triton mass value by using the conservation laws much as a surveyor finds the length of one side of a polygon from the other measurements of the polygon by using Euclidean geometry* (Fig. 93). There is one important change. In physics the geometry is to be understood as Lorentzian. Thus we have

$$(m_3)^2 = (E \text{ component of AB})^2 - (p \text{ component of AB})^2$$

In this formula the energy and momentum components of AB are found from the energy and momentum components of the other three sides of the polygon; that is, of the other three particles. How does one get the $E$ and $p$ values for one of these particles; for example, for the bombarding deuteron? Answer: By a procedure (Fig. 94) rather different from that normally used in surveying a plot of land! Suppose that a surveyor were required to use a method analogous to the way of measurement used in the $H^2 + H^2 \longrightarrow H^1 + H^3$ experiment. He could do so only if he employed the following unusual procedure to determine the north-south and east-west components of a boundary line CB (Fig. 94, after translation from the language of particle physics to the language of surveying!): (1) Measure the length of the line CB. (2, 3) Measure its component along the north-south direction. (4) Apply the Pythagorean theorem to evaluate the east-west component of the line CB.

We have now outlined how the components of the momentum-energy vectors of the target deuteron (OC, Fig. 93), the bombarding deuteron (CB), and the emerging proton (OA), are found. The components of the fourth, unknown, leg of the polygon (AB: the triton) are then found by a simple combination of the other three, known, 4-vectors, a calculation glorified by the phrase "application of the law of conservation of momentum and energy;" thus

$$\bar{p}_3{}^k = p_2{}^k + p_0{}^k - \bar{p}_1{}^k \qquad (k = x, y, z, t)$$

where the subscript zero refers to the initial deuteron at rest. The magnitude of this last leg gives at once the desired mass,

$$(m_3)^2 = (\bar{p}_3{}^t)^2 - (\bar{p}_3{}^x)^2 - (\bar{p}_3{}^y)^2 - (\bar{p}_3{}^z)^2$$

*Every application of conservation laws refers to polygon built of 4-vectors*

The determination of mass from the reaction $H^2 + H^2 \longrightarrow H^1 + H^3$ has been seen from this review to be highly geometric in character. The example illustrates a general principle. *Every application of the law of conservation of momentum and energy is a statement about a polygon built of 4-vectors in spacetime.* Apart from the difference between Lorentz geometry and Euclidean geometry, the arithmetic is no different from what one uses in surveying, in trigonometry, and in every other analysis of triangles and polygons. This comparison of particle physics and surveying suggests better than can be indicated in any other way the variety of situations that one encounters in the analysis of experiments. There is no problem connected with collisions, reactions, and transformation processes that does not have an analog in elementary geometry. A few examples of such problems and their analogs have been selected and are listed in Table 12.

To give prescriptions for analyzing all the types of collisions and transmutations which can and do arise in physics is out of place here even as it would be inappropriate in a brief text on the key ideas of Euclidean geometry to list and solve all the multitude of problems that can arise there! The character of the typical problem may be described by generalizing from the analogies in Table 12: *Given* such and such sides of a polygon, plus such and such north-south and east-west and up-down projections, plus such and such angles; *predict* this or that length ("rest mass"), projection ("energy or momentum"), or angle ("velocity relative to another particle or relative to laboratory"). To go into the varied arithmetic of great numbers of such problems would not illuminate the basic principles. For particle physics, those "principles" come back in the end to two very simple features of spacetime geometry: (1) The vector sum of the momentum-energy 4-vectors of all particles involved in a reaction is zero (provided that the 4-vectors of the *products* of the reaction are treated as having reversed sign). (2) The invariant magnitude of each 4-vector is equal to the rest mass of the particle under consideration.

*Conservation laws applied to analysis of collisions and transmutations: knowns vs unknowns*

In the application of these ideas one is guided by the standard principles of algebra. (1) To find $n$ distinct unknowns one must have $n$ independent equations in which all other quantities are known. (2) If only $n - r$ independent equations are available, then $r$ of the unknown quantities will be indeterminate. (An example is the collision of a deuteron of specified energy with a deuteron at rest, to produce a triton and a proton. Even when the rest masses of all four particles are given, it is impossible to predict the outcome of the reaction. The reason is simple. The proton may come off in any one of an infinitude of directions, at its own pleasure. The *angle of emission is indeterminate* in this problem. If one gives the angle as a separate piece of information ($\theta = 90°$ in the example) he can predict the energy. Conversely, if he gives the energy, he can predict the angle.) (3) If $n + s$ independent equations are available to predict $n$ unknowns, then the first $n$ equations suffice to determine the unknowns. The remaining $s$ equations serve as *checks* on the accuracy of the measurements or the correctness of the physics or both. In applying these principles one often takes the primary quantities to be the component values $E$ and $p^x$ and $p^y$ and $p^z$ of the several particles, both as a matter of bookkeeping convenience, and as a systematic way to count the number of knowns and unknowns.

An example of how one counts off knowns and unknowns is furnished by the (deuteron) + (deuteron) $\longrightarrow$ (proton) + (triton) reaction, regarded as a means to determine the mass of the triton. Table 13 examines this example.

**Table 12.** Finding a mass or an energy or another physical quantity by use of the conservation laws is analogous to finding the length of one side of a polygon or an angle or another geometric quantity by use of Euclidean geometry.

| *Particle physics* | | *Analog in Euclidean geometry* |
|---|---|---|
| *Process* | *Problem* | |
| A(fast) + B(target) ⟶ C(observed) + D(undetected) | *Known:* $m_A$, $m_B$, $m_C$.<br>*Measure:* $E_A$ and $E_C$ and direction of $p_C$ relative to $p_A$.<br>*Calculate:* unknown mass $m_D$. | *Given* (for an irregular polygon of four sides which does not lie in a plane): lengths of three sides, north-south components of these three sides, and angle between two of these sides as seen by a person sighting along third side.<br>*Find:* length of fourth side. |
| photon (momentum $p$) + electron (at rest) ⟶ electron (set in motion) + photon (momentum $\bar{p}$) | *Given:* rest mass $m$ of electron, initial momentum (or energy, $E = p$) of photon, and direction of emergence of the final photon.<br>*Predict:* momentum $\bar{p}$ (or energy, $\bar{E} = \bar{p}$) of that photon ("Compton effect," see Ex. 70). | *Given* (for an irregular polygon of four sides which does not lie in a plane): lengths of all four sides, north-south components of two sides (analogous to "energy of photon and electron before collision!"), and the angle between two of these sides ("photon before and after") as seen by a person sighting along third side ("target electron").<br>*Find:* east-west component of one unknown side. |
| ${}_{94}Pu^{239}$ (at rest) ⟶ ${}_{56}Ba^{144}$ + ${}_{38}Sr^{95}$ (spontaneous fission into exactly two fragments) | *Measure:* velocities of heavy and light fragments by time-of-flight experiment; also, mass of $Pu^{239}$ by mass spectrometer.<br>*Find:* rest masses of both fragments. | *Given:* long side of a triangle ("plutonium rest mass") and the two adjacent angles ("velocity parameters $\theta$ from velocities $\beta = \tanh \theta$").<br>*Find:* other two sides. |
| Same process as in previous example. | *Given:* data from measurements in previous example.<br>*Find:* kinetic energy set free in fission. | *Given:* data given in previous example.<br>*Find:* difference between long side and sum of other two sides. |
| $\mu$ (mu-meson at rest) ⟶ $e$ (fast electron) + $\nu$ (neutrino; speed of light)<br>(spontaneous decay of mu-meson in $\sim 10^{-6}$ second) | *Known:* rest mass of electron.<br>*Measure:* kinetic energy of electron ejected in this transmutation process.<br>*Find:* rest mass of mu-meson. | *Known:* shorter two sides of a triangle ("rest mass $m$ of electron and rest mass 0 of neutrino") and one angle ("velocity parameter $\theta$ of electron from its energy $E = m \cosh \theta$").<br>*Find:* long side of triangle. |

**Table 13.** Count of knowns and unknowns in the reaction $H^2 + H^2 \longrightarrow H^1 + H^3$.

*Commentary:* It is assumed, as in the text, that one does not have available a mass-spectrometer value of the mass of $H^3$ at the time when he determines this mass from the balance of momentum and energy in the reaction. Quantities which have been measured are marked YES in the table; those which have not, are marked NO.

For each of the four particles there are five symbols (four components and the mass), a total of twenty quantities. Of these ten are known (marked YES in the table), and ten are unknown. There are exactly ten equations to use in finding these ten unknowns. Therefore, it is not surprising that one can combine the information in these ten equations to obtain a unique equation (Eq. 99) for the desired triton mass $m_3$ in terms of measured quantities.

| | | $E = p^t$ | $p^x$ | $p^y$ | $p^z$ | *Invariant magnitude of 4-vector* |
|---|---|---|---|---|---|---|
| Reactants (list all components of 4-vector of momentum and energy with positive sign in table) | $H^2$ (target) | NO (measured $m_2$, not $E_2$ directly) | YES (zero!) | YES (zero) | YES (zero) | $m_2$ YES (Eq. 100) (spectrometer) |
| | $H^2$ (fast) | NO (measured *KE*; see below) | NO | YES (zero) | YES (zero) | $m_2{}^* = m_2$ YES (spectrometer) |
| Products of reaction (list all components with negative sign) | $H^1$ (measured) | NO (measured *KE*: see below) | YES (zero) | NO | YES (zero) | $m_1$ YES (Eq. 101) (spectrometer) |
| | $H^3$ (not measured) | NO | NO | NO | NO | $m_3$ "NO" (Eq. 105) (to be *found*) |
| Sum, gives change in total energy-momentum 4-vector for system; must vanish in order for 4-vectors to form closed polygon ("conservation law") | | 0 (Eq. 94) | 0 (Eq. 95) | 0 (Eq. 96) | 0 (Eq. 97) | |
| | | YIELDS FOUR EQUATIONS | | | | |

*Supplementary information. YIELDS SIX EQUATIONS:*

$E_2{}^* - m_2{}^* = 1.808$ MeV (kinetic energy of bombarding deuteron) (Eq. 102)

$E_2 - m_2 = 0$ (kinetic energy of target deuteron—assumed to be at rest)

$\bar{E}_1 - m_1 = 3.467$ MeV (kinetic energy of product proton) (Eq. 103)

$\bar{E}_3{}^2 - \bar{p}_3{}^2 = m_3{}^2$ (momentum-energy 4-vector for product triton) (Eq. 98)

$\bar{E}_1{}^2 - \bar{p}_1{}^2 = m_1{}^2$ (momentum-energy 4-vector for product proton)

$(E_2{}^*)^2 - (p_2{}^*)^2 = (m_2{}^*)^2 = m_2{}^2$ (momentum-energy 4-vector for incident deuteron)

"Rest mass can be transformed into energy and energy can be transformed into rest mass"—this is a loose way to summarize some consequences of the two principles that are basic and really accurate: (1) the total momentum-energy 4-vector of the system is unchanged in a reaction; and (2) the invariant magnitude of the momentum-energy 4-vector of any given particle is equal to the rest mass of that particle. How much sound information about physics can be extracted from these basic principles? What troubles sometimes arise from accepting too loose a formulation of the "principle of equivalence of mass and energy"? Some answers to these two questions are given in Table 14.

Table 14

## USES AND ABUSES OF THE CONCEPT OF MASS

| | |
|---|---|
| **Does *rest mass* have the same value in every inertial frame?** | Yes. Given in terms of energy $E$ and momentum $p$ by $m^2 = E^2 - p^2$ in one frame, by $m^2 = (E')^2 - (p')^2$ in another frame. Rest mass is thus an *invariant*. |
| **Does *energy* have the same value in every inertial frame?** | No. Energy is given by $E = (m^2 + p^2)^{1/2}$ or $E = m \cosh \theta = m/(1 - \beta^2)^{1/2}$ or $E = (\text{rest mass}) + (\text{kinetic energy}) = m + T$; value depends upon frame of reference from which particle (or system of particles) is viewed. Value is lowest in frame of reference in which particle (or system of particles) has zero momentum (zero *total* momentum in case of a system of particles). Only in that frame of reference is energy equal to rest mass. |
| **Is energy zero for an object of zero rest mass? (photon: light quantum; x-ray; gamma ray).** | No. Energy has value $E = (0^2 + p^2)^{1/2} = p$ (or in conventional units $E_{con} = c\,p_{con}$). Alternatively one can say—formally—that entire energy resides in the form of *kinetic* energy ($T = p$ in this special case of *zero* rest mass), none at all in the form of rest energy; thus, $E = (\text{rest mass}) + (\text{kinetic energy}) = 0 + T = T = p$ (case of zero rest mass only!) |
| **Does the invariance of rest mass mean that rest mass cannot change in a collision?** | No. Rest mass often changes in an *inelastic* encounter. Example 1: Collision of two balls of putty—hotter and therefore very slightly more massive after collision than before. Example 2: Collision of two electrons ($e^-$) with sufficient violence to produce a new pair consisting of one ordinary electron and one positive electron ($e^+$): $e^-$ (fast) $+ e^-$ (at rest) $\longrightarrow e^+ + 3e^-$. |
| **How can a quantity be *invariant* and yet *change* as a consequence of a collision?** | Invariance means "same value as determined from different inertial frames of reference," *not* "unchanged by impact or external forces." |
| **Does rest mass change in *every* inelastic encounter?** | No. Example: In the collision $e^-\text{ (fast)} + e^-\text{ (at rest)} \longrightarrow 2\begin{pmatrix}\text{electrons of} \\ \text{intermediate} \\ \text{speed}\end{pmatrix} + \begin{pmatrix}\text{electromagnetic energy} \\ \text{or } \textit{photons} \text{ emitted} \\ \text{in the collision process}\end{pmatrix}$ rest masses of individual electrons are the same after collision as before. |

**Does rest mass ever change in an *elastic* collision?**

No—by definition of elastic collision! Example:

$$e^-(\text{fast}) + e^-(\text{at rest}) \longrightarrow 2\begin{pmatrix}\text{electrons of}\\ \text{intermediate}\\ \text{speed}\end{pmatrix} + \begin{pmatrix}\textit{no}\text{ emitted}\\ \text{radiation}\end{pmatrix}$$

**Is the rest mass of a *system* composed of a number, *n*, of freely moving particles equal to the *sum* of the rest masses of the individual particles? Example: Box of hot gas.**

No. Rest mass $M$ of the system is greater than the sum of rest masses of particles—unless by chance all particles happen to be moving in same direction with a common speed. What is additive is not rest mass but *energy* and *momentum*:

$$E_{\text{system}} = \sum_{i=1}^{n} E_i$$

$$p^x{}_{\text{system}} = \sum_{i=1}^{n} (p^x)_i$$

From these sums the rest mass of the *system* can be evaluated:

$$M^2 = (E_{\text{system}})^2 - (p^x{}_{\text{system}})^2 - (p^y{}_{\text{system}})^2 - (p^z{}_{\text{system}})^2$$

**Does this relation simplify when the total momentum of the system is zero? *Example 1:* Box of hot gas at rest in laboratory.**
***Example 2:* *Any* system of particles in free motion when viewed from an inertial frame so chosen as to *make* the total momentum zero!**

Yes. In this case the rest mass of the *system* is given by the sum of energies of individual particles:

$$M = E_{\text{system}} = \sum_{i=1}^{n} E_i$$

Moreover, energy of each particle can always be expressed as sum of rest energy and kinetic energy:

$$E_i = m_i + T_i \qquad (i = 1, 2, \ldots, n)$$

So rest mass of a *system* exceeds sum of rest masses of its individual particles by an amount equal to the total kinetic energy of all particles (as seen in frame in which *total* momentum is zero!):

$$M = \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} T_i$$

**Has the "rest mass of the *system*" any *experimental* significance?**

Yes. Rest mass of the system determines its inertia: resistance to acceleration by a force that acts on the system as a whole. (Example: A box of hot gas offers, in principle, more resistance to acceleration than the same box after the gas is cooled.) Rest mass of the system also governs the gravitational attraction that it exerts on a test particle. (Example 1: A hot star containing specified numbers and kinds of atoms exerts, in principle, more pull on a distant planet than does the same collection of atoms after cooling. Example 2: A cloud of electromagnetic radiation consists of photons, each with zero rest mass but with positive "kinetic energy." Therefore, rest mass of cloud of radiation is positive. It exerts a gravitational attraction on a distant object, such as a sun, and in turn responds to gravitational attraction of that sun.)

**Does the explosion in space of a 20-megaton hydrogen bomb convert 0.93 kilogram of *mass* into *energy*? [$\Delta m = \Delta E/c^2 = (20 \times 10^6$ tons) $\times$ ($10^6$ grams/ton) $\times$ ($10^3$ calories/gram of "TNT equivalent") $\times$ (4.18 joules/calorie)/$c^2$ = ($8.36 \times 10^{16}$ joules)/($9 \times 10^{16}$ meters$^2$/second$^2$) = 0.93 kilogram].**

Yes *and* no; question needs to be stated more carefully. Rest mass of the system of expanding gases, fragments, and radiation has *same* value after explosion as before; rest mass $M$ of system has *not changed*. However, hydrogen has been transmuted to helium and other nuclear transformations have taken place. In consequence the *makeup* of rest mass of the system,

$$M = \sum m_i + \sum T_i$$

has changed. The first term on the right—sum of rest masses of individual constituents—has *decreased* by 0.93 kg:

$$(\sum m_i)_{\text{after}} = (\sum m_i)_{\text{before}} - 0.93 \text{ kilogram}$$

The second term—sum of kinetic energies, including "kinetic energy" of photons and neutrinos that are produced—has *increased* by the same amount:

$$(\sum T_i)_{\text{after}} = \underbrace{(\sum T_i)_{\text{before}}}_{\substack{\text{original heat content}\\ \text{of bomb—practically}\\ \text{zero by comparison}\\ \text{with 0.93 kg}}} + 0.93 \text{ kilogram}$$

Thus part of the rest mass of *constituents* has been converted into energy; but the rest mass of the *system* has not changed.

**Is the mass of the products of a nuclear explosion—contained in an underground cavity, allowed to cool, collected, and weighed—less than the mass of the original nuclear device?**

Yes. Key point is the waiting period, which allows heat and radiation to flow away until transmuted materials have same heat content as that of original bomb. Then in the expression for the rest mass of the system

$$M = \sum m_i + \sum T_i$$

the second term—whose value rose suddenly at time of explosion, but dropped during cooling period—has undergone no net alteration as a consequence of the explosion followed by cooling. In contrast, the sum of rest mases, $\sum m_i$, has undergone a permanent decrease; and with it, the mass $M$ of *what one weighs* (after the cooling period) has dropped (Fig. 95).

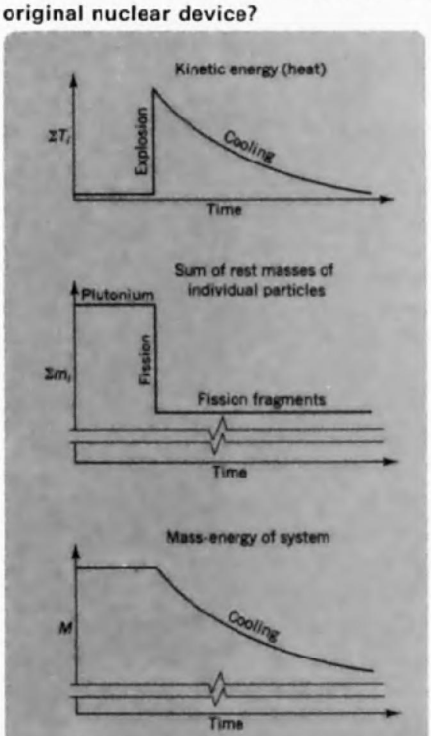


**Fig. 95.** Total kinetic energy, sum of rest masses of individual particles, and rest mass of system as functions of time when nuclear device explodes and products are allowed to cool.

**Does Einstein's statement that mass and energy are equivalent mean that energy is the *same* as mass?**

No. Value of energy depends upon inertial frame of reference from which particle (or system of particles) is regarded. Value of rest mass is independent of inertial frame. Energy is only the time *component* of a 4-vector, whereas mass measures entire *magnitude* of that 4-vector (see also Ex. 67). The *time component* gives the *magnitude* of the 4-vector only in the special case in which that 4-vector has no space component; that is, when the momentum of the particle (or the total momentum of the system of particles) is zero. Only in this special case does energy have the same value as rest mass.

**Without delving into all fine points of legalistic phraseology, is the equation $E_{\text{conv}} = mc^2$ what is *really* significant about the equivalence of mass and energy?**

Historically, yes; today, no! In earlier days one did not recognize that joules and kilograms are two units—different only because of historical accident—for one and the same kind of quantity, mass-energy. Similarly ergs and grams are different units for the single quantity, mass-energy. The conversion factor $c^2$, like the factor of conversion from seconds to meters or miles to feet, can today be counted, if one wishes, as a detail of convention, rather than as a deep new principle.

**If the factor $c^2$ is not the central feature of the relationship between mass and energy, then what *is* central?**

The distinction between mass and energy is this: mass measures the magnitude of a 4-vector and energy measures the time component of the *same* 4-vector. Any feature of any discussion that emphasizes this contrast is an aid to understanding. Any slurring of terminology that obscures this distinction is a potential source of error or confusion.

**The rest mass $M$ of a system of freely moving particles is given, not by the sum of the *rest masses* $m_i$ of the individual constituents, but by the sum of the *energies* $E_i$ *(but only in a frame in which the total momentum of the system is zero)*. Then why not give $E_i$ a new name and call it the "relativistic mass" of the individual particle? With this notation**

$$m_{i,\,\text{rel}} = E_i \begin{cases} = m_i + T_i \\ = (m_i^2 + p_i^2)^{1/2} \\ = m_i/(1 - \beta_i^2)^{1/2} \end{cases}$$

**one can then write**

$$M = \sum_{i=1}^{n} m_{i,\,\text{rel}}$$

The concept of "relativistic mass" is subject to misunderstanding and is not used here. (1) It applies the name "mass"—belonging to the *magnitude* of a 4-vector—to a very different concept, the *time component* of a 4-vector. (2) It makes increase of energy of a particle with velocity or momentum appear to be connected with some change in internal structure of the particle. In reality increase of energy with velocity originates in geometric properties of spacetime itself (Lorentz transformation!).

**Can any simple diagram illustrate adequately this contrast between mass and energy?**

Yes! Figure 96 shows the momentum-energy 4-vector of the same particle as seen in different reference frames. Energy differs from frame to frame. Rest mass (magnitude of 4-vector) has same value $m$ in all frames. (Apparent difference between values of $m$ in the three frames comes about because of trying to represent Lorentz geometry on the Euclidean page. In Lorentz geometry, square of hypotenuse is difference between squares of $E'$ and $p'$, or $E''$ and $p''$.)

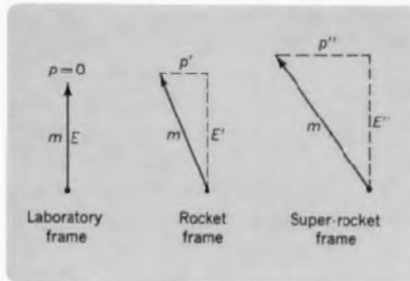


**Fig. 96.** Momentum-energy 4-vector of the same particle as seen in three different reference frames.

**Is there any equally simple diagram to illustrate the transformation of a part of the rest mass of a plutonium nucleus into energy in the process of fission?**

Yes! Figure 97. The vector sum of two timelike 4-vectors has magnitude $M$ (rest mass of $Pu^{239}$ before fission), which is greater than sum of magnitudes, $m_1$ and $m_2$, of the two individual 4-vectors (rest masses of fission products). Contrasts to Euclidean geometry in which the third side of a triangle always has a length that is less in value than sum of lengths of other two sides.

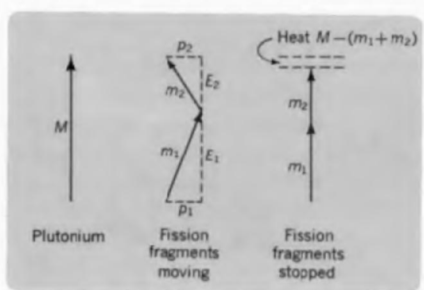


**Fig. 97.** The sum of the rest masses of the fission fragments of plutonium is less than the rest mass of the original plutonium nucleus.