Momentum and Energy

2

10. Introduction; Momentum and Energy in Units of Mass

Physics is concerned with matter and motion and the forces that cause motion. What then is the relation between force and motion? It is not necessary in this brief account to catalogue electric and magnetic and all the other kinds of force. Instead, we ask a more urgent question. How can one tell whether any force at all is acting on a particle? And, if a force is acting on a particle, what feature of the world line of that particle reveals the presence of this force? And finally, how do changes in the energy and momentum of the particle measure the strength of this force?

Momentum change of one object as symptom of interaction with another object

To understand the nature of the concept, "force," try to imagine how one could get along without it! Force is most obviously needed to explain why a particle speeds up or slows down. A test particle, subject to no forces, is defined precisely by the fact that it does not speed up or slow down. Relative to an inertial frame it continues to be in a state of rest or in motion at constant velocity. It traces out a straight world line. This world line appears in Fig. 79 for the special case of a particle moving in the x direction. In contrast, Fig. 80 plots the world line of a particle that evidently *does* change its velocity and therefore *is* subject to a force. No one has ever found such a change in velocity to take place without something to cause it—typically either a collision with a nearby particle or a force caused by a distant particle. Therefore, force can be said to be a form of *interaction* (Fig. 81). This way of speaking is further justified by two additional circumstances: (1) When the presence of A brings about a change in the velocity of B, the presence of B also causes an alteration in the velocity of A. (2) After the interaction has ended and the particles have sepa-

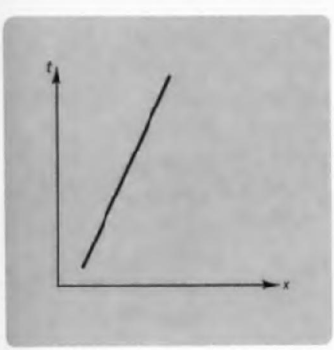


Fig. 79. World line of a particle subject to no forces.

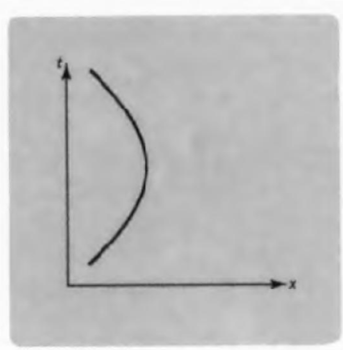


Fig. 80. World line of a particle subject to a force.

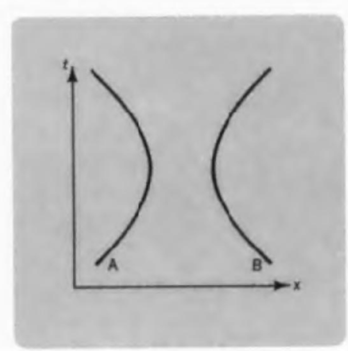


Fig. 81. World lines of two interacting particles.

rated, the change of *momentum* of one particle is equal in magnitude and opposite in direction to the change of momentum of the other particle. Thus, instead of talking about forces between particles, we can talk about their changes of momentum. Indeed to discuss both momentum *and* forces in the context of relativity is to complicate the story. In this account we analyze momentum alone.

Momentum defined so it will be conserved

How is momentum to be defined? Early workers in Newtonian mechanics defined momentum as the mass of a particle times its velocity. What makes this definition useful is the fact that momentum so defined is conserved in low-velocity collisions. However, observations show that momentum defined, according to Newtonian principles, as mass times velocity is *not* conserved in high-velocity collisions. We must therefore choose: We must abandon *either* the Newtonian expression for momentum *or* the law of conservation of momentum. The law of conservation of momentum has become so important to us that we shift to it as a new foundation. We *start* with the law of conservation of momentum and from it *derive* the expression for *momentum defined as that vector quantity which is conserved in all frames of reference*.

This requirement that momentum be conserved in all frames of reference will be used *three* times in this chapter, and each time its use will produce a revolution in our way of looking at nature. In the next section the requirement will be applied to a glancing elastic collision in two dimensions to derive the *relativistic expression for momentum* of a particle. In Section 12 the conservation requirement will be applied to a one-dimensional collision to derive the *relativistic expression for energy* of a particle. In Section 13 the conservation requirement will be applied to an inelastic collision to derive the *equivalance of energy and rest mass*. One asks, how can the law of conservation of momentum be worthwhile if momentum and energy are so *defined* as to make the law hold true. This question takes us to the heart of the nature of the laws and theories of physics.† For an answer, look at an object that, like a billiard ball, bats about, colliding with one body after another. In analyzing the first collisions, we use the conservation law to find out or define the unknown momenta of the several objects. The situation is quite different in the subsequent collisions. The momenta in these collisions are already known. There the law of conservation of momentum is upheld, not by definition, but by the inner workings of the world's machinery. All the laws and theories of physics have this deep and subtle character, in that they both define for us the needful concepts and make statements about these concepts. Contrariwise, the absence of some body of theory, law, and principle deprives us of a means properly to use or even to define concepts. How far out of date is that view of science which used to say, "Define your terms before you proceed."! The truly creative nature of any forward step in human knowledge is such that theory, concept, law, and method of measurement—forever inseparable—are born into the world in union.

Multiple checks test that conservation law is not mere circular reasoning

Thus physics provides a way to *harmonize* the experimental facts. No single experiment suffices to establish a conservation law. At least two are needed—

†See Henri Poincare, *The Foundations of Science*, translated by G. B. Halsted (Science Press, Lancaster, Pennsylvania, 1946), pp. 310 and 333.

one to establish the definition of the conserved quantity, the other to verify that this quantity really is conserved. In this chapter we will be concerned with the first of these experiments, the one necessary to establish definitions. The *checking* of these definitions proceeds daily and hourly in the ongoing enterprise of experimental physics.

In Newtonian mechanics the momentum of a particle is defined as mass times velocity. In Chapter 1 we measured velocity β in units of meters of distance traveled per meter of light-travel time. In terms of this velocity the Newtonian expression for momentum is $m\beta$. This expression says nothing new about momentum (and is *not* the relativistic expression for momentum!), but makes it clear that time is measured in meters. *If time is measured in meters, momentum has the units of mass.* Conversion to conventional units (for instance, kilogram-meters per second) requires multiplication by the conversion factor c (the speed of light) to convert β to v, thus: (Newtonian momentum in conventional units) $= m\beta c = mv$.

Momentum and energy most conveniently expressed in units of mass

Similarly, in Newtonian mechanics the kinetic energy of a particle is defined as mass times velocity squared divided by two. In terms of velocity β measured in meters per meter, the Newtonian expression for kinetic energy is $1/2\ m\beta^2$. This expression says nothing new about energy (and is *not* the relativistic expression for energy), but makes it clear that time is measured in meters. *If time is measured in meters, energy has the units of mass;* and energy and momentum have the same units. Conversion to conventional units (for instance, joules) requires multiplication by the conversion factor c^2 (the square of the speed of light) to convert β^2 to v^2, thus: (Newtonian kinetic energy in conventional units) $= 1/2\ m\beta^2c^2 = 1/2\ mv^2$.

The symbols for momentum (p) and kinetic energy (T) in units of mass will be given without subscripts. Thus, in the Newtonian limit of low velocity,

$$\left.\begin{aligned} p &= m\beta \\ T &= 1/2\ m\beta^2 \end{aligned}\right\} \quad \text{(for low velocity—units of mass)} \tag{67}$$

On the other hand, the symbols for momentum and energy in conventional units will be given the subscript *con*—intentionally unwieldy to make the use of conventional units distasteful. Thus, in the Newtonian limit of low velocity,

$$\left.\begin{aligned} p_{con} &= mv \\ T_{con} &= 1/2\ mv^2 \end{aligned}\right\} \quad \text{(for low velocity—conventional units)} \tag{68}$$

In this chapter the relativistic expressions for momentum and energy will be derived in units of mass. Expressions for momentum and energy in units of mass are easily converted to conventional units by multiplication by c and c^2 respectively. A summary of results in both sets of units is tabulated inside the back cover of this book.

11. Momentum

How much can one deduce about momentum simply from what he knows about the structure of spacetime, in advance of any experiments? Specifically, if there exists for each particle such a vector quantity as the momentum, the sum of which for all particles is conserved when these particles interact, then how must the momentum of each particle depend upon its velocity? Momentum is a vector quantity, so we must first determine the *direction* of the momentum vector of a particle and second the way in which its *magnitude* depends upon the speed of the particle. We begin by reasoning that the momentum vector of a particle points in the direction in which the particle is moving. This conclusion can be supported by an *argument from symmetry*—a powerful tool in physics—in the following way. In an inertial frame, space is the same in all directions; we say that space is *isotropic*. Since this is so, the only unique direction associated with the motion of a particle that moves in a straight line is the direction in which the particle is moving. If the momentum vector of a particle did not point in the direction of motion but, say, at an angle of thirty degrees from this direction, then there would be a large number of vectors—all thirty degrees from the direction of motion, all of them equivalent—any one of which could represent the momentum. But space is isotropic. Therefore we could not choose any one of these vectors in preference to any other of these vectors. But we have assumed that the momentum is determined uniquely, both in magnitude and in direction, by the velocity. Thus we have come against a contradiction. There is only one escape. The line of the momentum vector must lie along the line of motion of the particle. Of the two possible directions along the line of motion of a particle we arbitrarily *choose*, as the direction of the momentum vector, the *same* direction as the velocity of the particle.† The conclusion, finally, is that *the momentum vector of a particle points in the direction of its velocity.*

Symmetry shows momentum is parallel to velocity

Momentum conservation law used to find how momentum depends on velocity

Now we know the *direction* in which the momentum vector of a particle points. The second problem is to find the *magnitude* of this momentum vector. We do this by requiring that total momentum be conserved in an elastic collision. This requirement—coupled with the invariance of the interval in Lorentz geometry—will be sufficient to show that the Newtonian expression for momentum

$$p = m\beta \quad (= m \tanh \theta) = m \times (\text{displacement per unit of time})$$

must be replaced by the relativistic formula

$$(69) \quad p = m \sinh \theta = m\beta/(1 - \beta^2)^{1/2} = m \times (\text{displacement per unit of proper time})$$

For small velocity β (small velocity parameter θ) the exact relativistic expression becomes approximately equal to the Newtonian expression.

Total momentum is zero before collision in suitably chosen frame

Let the colliding objects be two identical balls A and B and let the collision be not head-on (rare!) but glancing (typical). One can always find a reference frame moving at such a velocity that *before* the collision the two balls have equal and opposite velocities (Fig. 82). In this frame the *total* momentum of the two identical balls is zero.

†We *could*, of course, choose, as the direction of the momentum vector of a particle, the direction *opposite* to its direction of motion. Such a choice would be consistent with the symmetry of the problem and would lead to no physical contradiction if used for all particles. Under such circumstances the momentum of each particle and the total momentum of a system of particles would point in the direction opposite to the corresponding momenta defined above. It is customary to have the momentum vector of a particle lie in the *same* direction as its velocity.

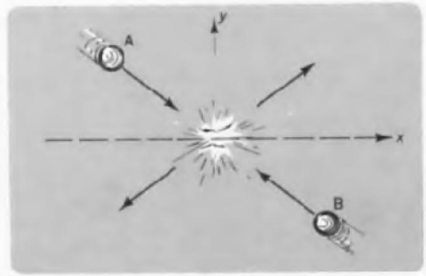


Fig. 82. Glancing elastic collision as observed in a frame moving in such a way that *before* collision the two balls have equal speeds but move in opposite directions. Argument in the text shows that *after* elastic collision the two balls again move with their original speed and again move in opposite directions as observed in that same frame.

The conclusion that the total momentum is zero can be confirmed by the following symmetry argument: Suppose the total momentum is not zero in this "velocity-symmetric" frame. Then a contradiction will arise, as spelled out in the following argument. A second pair of balls started off exactly as in Fig. 82—except with B located where A is in the figure, and A located where B is—will present no new features. Therefore the total momentum will have the same magnitude, and point in the same direction, as does the total momentum in Fig. 82 (not shown because in reality this total momentum is zero!) But the picture of this new collision can be derived from Fig. 82 by turning the book upside down (rotation of 180° in its own plane!). This operation reverses the direction of the total momentum. Consequently the total momentum vector must be identical to the total momentum vector as rotated through 180°! This contradiction can be resolved only if the total momentum vector has zero magnitude. Thus *before* the collision the two identical particles have equal and opposite momenta.

What happens *after* the collision? The balls must then move in opposite directions and with equal speeds. If they did not, their momenta would not add up to zero and total momentum would not be conserved in the collision—contrary to our requirement. Consider (for the present only) collisions that are *elastic* according to the following prescription: Take a motion picture of the collision. Run it backwards. Nothing will be changed about the collision except this: Particle A now passes from right to left, particle B from left to right instead of the other way around. In this sense an elastic collision is one that is *reversible*. If the collision pictured in Fig. 82 is elastic in this sense, then each ball changes its direction of motion but not (except temporarily) its speed; the overall effect of the collision is simply to rotate the velocity vectors of both particles. The x direction of the reference frame can be chosen to lie in such a direction that the x component of velocity of each particle does not change in the collision, and the y component of velocity of each particle reverses direction, as shown in the figure.

Appearance of collision in three different frames

We wish to study the total y component of momentum and its conservation in this collision. To do this most simply, observe the collision in a frame in which ball A moves only in the y direction. This is a rocket frame that moves to the right relative to the frame of Fig. 82 with the same velocity as the x component of velocity of ball A. The collision as observed in this rocket frame is shown in Fig. 83.

There is also a frame in which *ball B* moves only in the y direction. This is a laboratory frame that moves to the left relative to the frame of Fig. 82 with the same velocity as the x component of velocity of ball B. The collision as observed in this laboratory frame is shown in Fig. 84.

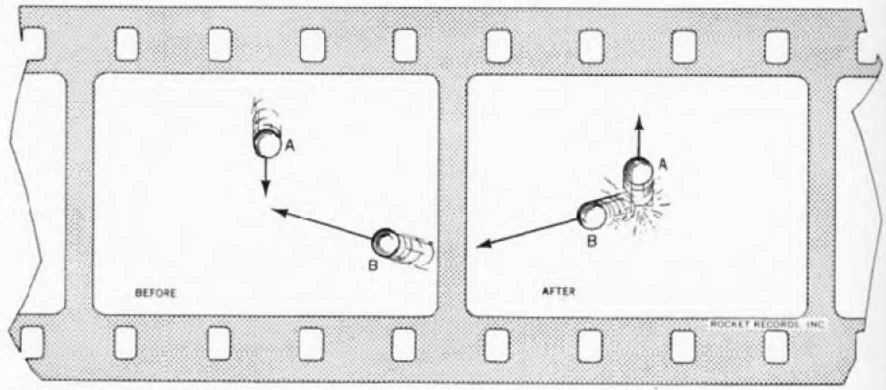


Fig. 83. Collision of Fig. 82 observed in rocket frame.

Our objective is to find out about the momentum of a particle—which may have a velocity very close to the speed of light—from what we know from Newtonian physics about the momentum of a particle of very low velocity. For this purpose a glancing collision is ideal. We can so arrange the encounter that the struck particle has arbitrarily low velocity not only before the collision but also afterwards (particle B in Fig. 84). Therefore the momentum of the struck particle can be obtained from the Newtonian formula $p = m\beta$ as well after the collision as before it. This circumstance will make it easy to determine the momentum change of the slow particle (B) in the collision and will thus —by the use of the law of conservation of momentum—allow us to get at the momentum change, and even the momentum, of the fast particle (A). Granted the symmetry of the diagram, one knows that the momentum transferred to B is twice its momentum before it is struck; or

$$1/2\ (\text{momentum change of B}) = m\ dy/dt$$

Momentum proportional to displacement per unit of proper time

Particle A transfers momentum to B, not by any alteration in the magnitude of its own momentum, but by a change in the *direction* of its momentum vector. In other words, the momentum transfer is the short and known side of a momentum triangle. The other two (equal) sides of the triangle are long and unknown. But the long and short sides of a similar triangle are known—the triangle of displacements. The proportional relation between the sides of similar

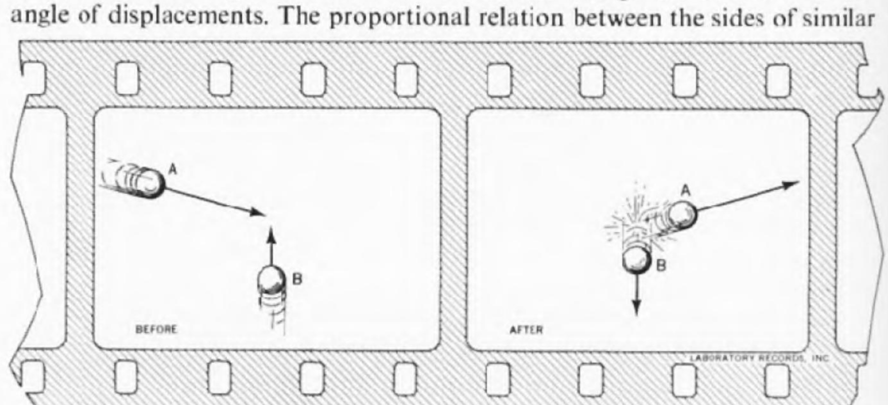


Fig. 84. Collision of Fig. 82 observed in laboratory frame.

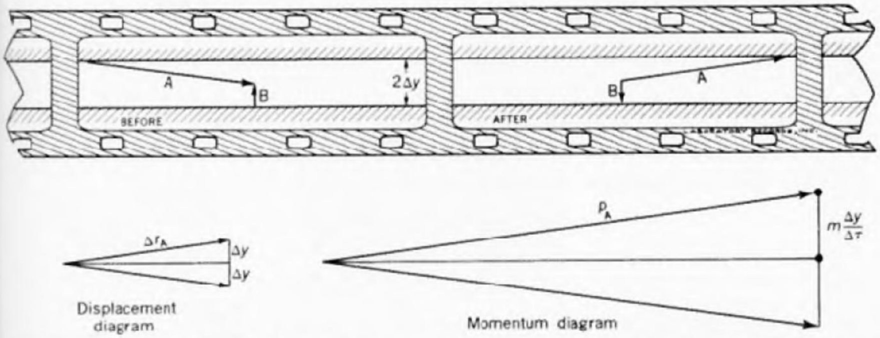


Fig. **85.** Derivation of relativistic expression for momentum from the law of conservation of momentum as applied to a glancing collision. Particle B is moving so slowly that the Newtonian expression for momentum gives an arbitrarily good approximation to its momentum: (momentum) $= m(\Delta y_B/\Delta t_B)$. Here Δt_B is the time required for the flight of particle B through the distance Δy_B from the lower boundary to the point of collision. This laboratory time is arbitrarily close in value to the *proper* time, $\Delta\tau_B$, for the flight—again because the velocity of B can be chosen to be arbitrarily slow. (Example: $\Delta\tau$ and Δt differ by 5 parts in 100,000 for $\beta = 0.01$) Therefore the momentum of B can also be written in the form $m(\Delta y_B/\Delta\tau_B)$. Knowing the momentum of B one can find the momentum p_A of A by comparing the momentum diagram above with the diagram for the displacement of A ("similar triangles"). The y displacement of A has been adjusted to be the same as the y displacement of B (symmetric positioning of the "floor" and "ceiling" on which B and A, respectively, impact): $\Delta y_A = \Delta y_B = \Delta y$. Also *the proper time from collision to impact is the same for A as for B*: $\Delta\tau_A = \Delta\tau_B$.

[PROOF: (1) The motion of A in the rocket frame is identical to the motion of B in the laboratory frame (compare Figs. 83 and 84). Consequently these *proper times* of flight are equal:

$$(\Delta\tau_A)_{\text{rocket frame}} = (\Delta\tau_B)_{\text{laboratory frame}}$$

(2) But the *proper* time between two events (collision and impact) has the same value in all frames of reference; or

$$(\Delta\tau_A)_{\text{laboratory frame}} = (\Delta\tau_A)_{\text{rocket frame}}$$

Consequently (3)

$$(\Delta\tau_A)_{\text{laboratory frame}} = (\Delta\tau_B)_{\text{laboratory frame}}$$

as was to be shown. Of course the *laboratory* clock indicates very different durations for the flights of A and B when A moves very close to the speed of light:

$$(\Delta t_A)^2_{\text{laboratory frame}} = (\Delta\tau_A)^2_{\text{laboratory frame}} + (\Delta x_A)^2_{\text{laboratory frame}} \gg (\Delta\tau_A)^2_{\text{laboratory frame}}$$
$$= (\Delta\tau_B)^2_{\text{laboratory frame}} = (\Delta t_B)^2_{\text{laboratory frame}}]$$

Therefore the momentum of A is at last known directly in terms of quantities which refer exclusively to the motion of A:

$$p_A = m(\Delta r_A/\Delta\tau_A)$$

Translating from finite differences to a derivative, and recalling that momentum has the same direction as displacement, one obtains the vector equation

$$\boldsymbol{p} = m\, d\boldsymbol{r}/d\tau$$

This is the relativistic formula for momentum, valid for a particle of arbitrarily great energy.

triangles yields at once (Fig. 85) an expression for the momentum of the high energy particle A

$$p = m\, dr/d\tau \qquad (70)$$
$$= m \text{ times displacement per unit of proper time}$$

The individual components† of this vector are

$$\begin{aligned} p^x &= m\, dx/d\tau \\ p^y &= m\, dy/d\tau \\ p^z &= m\, dz/d\tau \end{aligned} \qquad (71)$$

in the laboratory frame of reference.

In the rocket frame of reference the components of the momentum are given by expressions similar to Eqs. 71, except that they contain dx', dy', and dz', the components of displacement as measured in the rocket frame. The lapse of proper time, $d\tau'$, between two nearby events on the track of the particle has the same value when calculated from rocket measurements as when calculated from laboratory measurements ("invariance of the interval"). Therefore there is no need to distinguish $d\tau'$ from $d\tau$. Moreover, the value of dy' (rocket frame) is the same as the values of dy (laboratory frame); similarly, $dz = dz'$. Consequently *the components, $p^y = m\, dy/d\tau$ and $p^z = m\, dz/d\tau$, of the momentum transverse to the relative motion of the rocket and laboratory frames are independent of that relative motion.*

Momentum is analogous to displacement in this respect, that transverse components of both vectors are unaffected by the motion of the observer. There is a simple reason for this similarity in properties of the two vectors! The momentum is obtained from the displacement (Δx, Δy, Δz) by multiplication with a factor ($m/\Delta\tau$) that has the same value in all inertial frames.

Mass most usefully defined as the velocity-independent factor in the momentum

From the analysis of momentum in Fig. 85 it was clear that the quantity m is the mass as mass is understood in Newtonian physics. Thus *m is a constant*, the same at all speeds, all places, and all times. Any difference between the spacetime formula for momentum (for example, $m\, dx/d\tau$) and the corresponding Newtonian formula ($m\, dx/dt$) is therefore to be attributed to the difference between proper time and laboratory time, not to any difference in the value of m in the two descriptions of nature. In some older treatments the Newtonian expression for momentum ($m\, dx/dt$) was corrected, not by changing dt to $d\tau$—as today is considered the simple method—but by introducing a "modified mass" depending upon velocity in such a way that one is justified in still using the Newtonian type of formula; thus,

$$p^x \text{ (relativistic)} = m_{\text{modified}}\, (dx/dt)$$

This modified mass has to have the value

$$m_{\text{modified}} = m\, (dt/d\tau) = m/(1 - \beta^2)^{1/2} \qquad (72)$$

This notation is still used from time to time. However, the most useful quantities in the analysis of physics are often those, such as m and $d\tau$, that have the

†Why p^x and not p_x? In the four-dimensional geometry of spacetime—unlike the Euclidean geometry of space—the position of the label is important (details on standard notation are included in the footnote on page 118).

same value in all frames of reference. Today this fact is more and more widely recognized. Therefore *we will normally understand by the term "mass" the velocity-independent quantity m.*

Relativistic momentum reduces to Newtonian value in low-velocity limit

How much difference is there between the relativistic and the Newtonian expressions for momentum? The relativistic expression for momentum must reduce to the Newtonian expression for low-velocity particles. A low-velocity particle moves much less than one meter of distance $d\mathbf{r}$ in one meter of time dt. Therefore the proper time $d\tau = [(dt)^2 - (dr)^2]^{1/2} = (1 - \beta^2)^{1/2}\, dt$ for any displacement $d\mathbf{r}$ of such a particle is very nearly equal to the time dt itself

$$d\tau \approx dt \qquad \text{(low-velocity particle)}$$

(agreement to 5 parts in 100,000 for $\beta = 0.01$; perfect agreement for $\beta \longrightarrow 0$). Under these circumstances the relativistic expression for momentum, $\mathbf{p} = m\,d\mathbf{r}/d\tau$, becomes identical with the Newtonian expression, $\mathbf{p} = m\,d\mathbf{r}/dt$, it being understood that the quantity m is the same in the two cases (invariant m!).

It is sometimes convenient to express the magnitude of the momentum in terms of the velocity parameter θ of the particle or in terms of the particle speed $\beta = \tanh\theta$; thus

$$\begin{aligned} p &= m\frac{dr}{d\tau} = m\frac{dr}{[(dt)^2 - (dr)^2]^{1/2}} = \frac{m(dr/dt)}{[1 - (dr/dt)^2]^{1/2}} = \frac{m\beta}{[1-\beta^2]^{1/2}} \\ &= \frac{m\tanh\theta}{[1 - \tanh^2\theta]^{1/2}} = \frac{m\tanh\theta}{\left[\dfrac{\cosh^2\theta}{\cosh^2\theta} - \dfrac{\sinh^2\theta}{\cosh^2\theta}\right]^{1/2}} = \frac{m\tanh\theta\cosh\theta}{[\cosh^2\theta - \sinh^2\theta]^{1/2}} = m\sinh\theta \end{aligned}$$

so that

$$(73) \qquad p = m\sinh\theta = \frac{m\beta}{(1-\beta^2)^{1/2}} \qquad \text{(relativistic expression for momentum in units of mass)}$$

In contrast the Newtonian expression for momentum has the form

$$(74) \qquad p = m\beta = m\tanh\theta \qquad \text{(Newtonian expression for momentum in units of mass)}$$

These two expressions for momentum differ by the factor $dt/d\tau = \cosh\theta = 1/(1-\beta^2)^{1/2}$. This factor measures the ratio between laboratory time and proper time as recorded by a clock moving with the particle. This is the time-dilation factor of Ex. 10. The presence of this factor in the relativistic formula for the momentum tells one that a particle can transport an arbitrarily great amount of momentum into a collision process if it travels arbitrarily close to the speed of light. One would not suspect this result if he were guided only by the incorrect, Newtonian, formula for momentum, $p = m\beta$, for m is a constant and β cannot exceed unity.

The expression for the momentum of a high-velocity particle thus differs importantly from the Newtonian prediction. However, the procedure for determining the mass of a new particle—involved in a collision process—differs in no point of principle between relativistic and Newtonian theory. The basic idea can be stated in various terms: (1) the principle of action and reaction, (2) the principle that connects the kick of a gun with the momentum of the bullet, (3) the law of conservation of momentum.

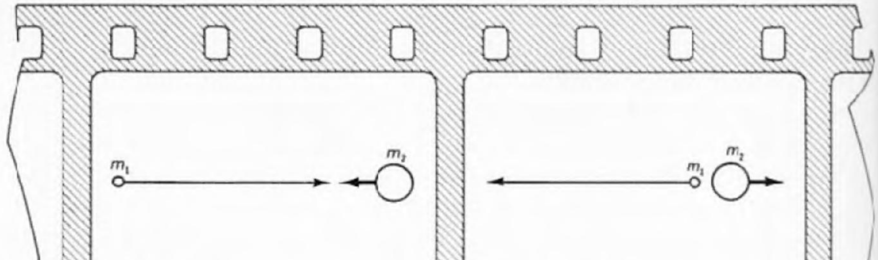


Fig. 86. Velocities before and after a head-on elastic collision as seen from the reference frame in which the total momentum is zero.

Mass of unknown particle determinable from elastic collision with a standard particle

Specifically, consider a head-on and elastic collision between (1) a standard particle, of mass m_1 (this mass arbitrarily assigned by an International Committee on Weights and Measures), and (2) the particle under investigation, with a mass m_2, which at present is unknown, but is to be determined. To say that the collision is head-on and elastic means that there exists a frame of reference in which the velocity record shows the before-after symmetry illustrated in Fig. 86. This symmetry implies that the total momentum reverses sign in the collision. But the total momentum stays constant in the collision. Therefore the total momentum must be zero. Consequently the momenta of the two particles individually, as observed after the collision, must satisfy the condition

$$m_1\,(dx/d\tau)_1 + m_2\,(dx/d\tau)_2 = 0$$

From this formula one deduces the value of the unknown mass in units of the known mass:

$$(75)\qquad m_2/m_1 = \frac{(-dx/d\tau)_1}{(dx/d\tau)_2} = \frac{-\Delta x_1}{[(\Delta t_1)^2 - (\Delta x_1)^2]^{1/2}}\,\frac{[(\Delta t_2)^2 - (\Delta x_2)^2]^{1/2}}{\Delta x_2}$$

Here Δx_1 and Δx_2 are the distances traveled by the two particles from the point of collision to the points of detection; and Δt_1 and Δt_2 are the respective times of flight. For elastic collisions at nonrelativistic velocities, the right-hand side of Eq. 75 reduces to the Newtonian value

$$(76)\qquad m_2/m_1 = -\beta_1/\beta_2 = \frac{-(\Delta x_1/\Delta t_1)}{(\Delta x_2/\Delta t_2)} \qquad \text{(Newtonian limit)}$$

The full simplicity of the relativistic concept of momentum does not become apparent until momentum is seen as the space part of a momentum-energy 4-vector. And only then does one see that checks of the balance of energy in collision processes serve as indirect tests of momentum conservation; tests that are added to all the myriad direct experimental tests of momentum conservation.

12. The Momentum-energy 4-vector

To see momentum and energy as parts of a larger unity it is helpful to recall how one sees space and time as parts of a larger unity. One considers the passage of a particle from an event A in spacetime to a nearby event B. The *4-vector* that leads from A to B is the unifying idea.† The components of this 4-vector, the displacements dx, dy, dz, and dt, have differing values according as the 4-vector is examined from one or another frame of reference. Despite this arbitrariness in the means that we use to describe the 4-vector AB, the 4-vector itself is well defined. Not only does the interval AB have the same magnitude in all frames of reference. More significantly, the *locations* of events A and B themselves—and therefore the positioning of the 4-vector AB in spacetime—are as well defined as the location of two town gates, regardless of what coordinates one uses, or even regardless of whether one uses any coordinates at all.

Energy as fourth component of momentum-energy 4-vector

In a similar way we expect to see the momentum and the energy of a particle at any given phase of its history as components—and merely components—of a 4-vector, which has a reality above all questions of coordinates. Moreover, the connection of this "energy-momentum 4-vector" with the displacement 4-vector AB is no remote and indirect one. Nothing could be more direct than the following chain of ideas:

(1) displacement 4-vector AB with components

$$dt,\ dx,\ dy,\ dz$$

illustrated in Fig. 87.

(2) unit tangent vector obtained by dividing the 4-vector AB by the proper interval $d\tau = [(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2]^{1/2}$ between one end of this displacement and the other; components

$$(dt/d\tau),\ (dx/d\tau),\ (dy/d\tau),\ (dz/d\tau)$$

are illustrated in Fig. 88.

†In 1872, in his inaugural lecture as professor at the University of Erlangen, Felix Klein initiated the decisive new point of view towards geometry which illuminates modern mathematics. His key idea distinguishes one kind of geometry from another by the *law of transformation of components*. One sees very clearly, for example, the difference between Euclidean geometry and the Lorentz geometry of the real physical world in the definition today employed for a vector:

> A *4-vector* is *defined* by giving in every inertial reference frame four numbers (numbers which differ from frame to frame!) such that these numbers transform from one frame to another by the Lorentz transformation formula (Eq. 32).

> A *3-vector* is *defined* by giving in every Euclidean coordinate system three numbers (components: numbers which differ from coordinate system to coordinate system!) such that these numbers transform from one coordinate system to another by the appropriate formula for Euclidean rotation (Eq. 29).

Given that a quantity is a vector, and knowing its components in only a single frame of reference, one can at once *deduce* its components in every other frame of reference from the appropriate three- or four-dimensional law of transformation of components.

(3) energy-momentum 4-vector, obtained by multiplying this unit vector by the constant m; components

(77)
$$\begin{aligned} E = p^t &= m(dt/d\tau) \\ p^x &= m(dx/d\tau) \\ p^y &= m(dy/d\tau) \\ p^z &= m(dz/d\tau) \end{aligned}$$

are illustrated in Fig. 89.

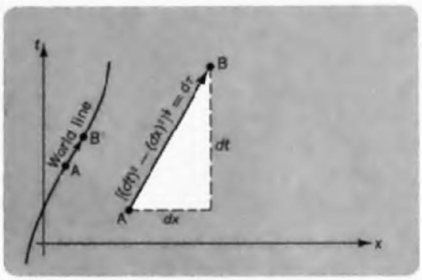


Fig. 87. Displacement 4-vector AB between two events A and B on the world line of a particle. Pictured here for the special case in which the y and z components of displacement dy and dz are both equal to zero.

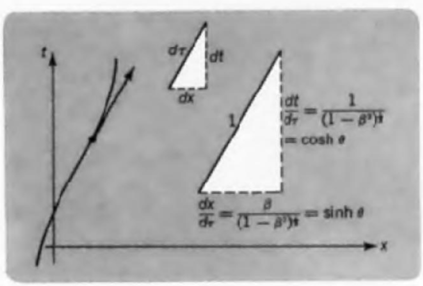


Fig. 88. Unit tangent vector to the world line of a particle. Obtained by dividing the displacement 4-vector AB of Fig. 87 by the invariant proper time interval $d\tau$. The time and space components of the unit tangent vector have the values

$$\frac{dt}{d\tau} = \frac{dt}{[(dt)^2 - (dx)^2]^{1/2}} = \frac{1}{[1 - (dx/dt)^2]^{1/2}} = \frac{1}{[1-\beta^2]^{1/2}} = \frac{1}{[1-\tanh^2\theta]^{1/2}}$$

$$= \frac{1}{\left[\frac{\cosh^2\theta}{\cosh^2\theta} - \frac{\sinh^2\theta}{\cosh^2\theta}\right]^{1/2}} = \frac{\cosh\theta}{[\cosh^2\theta - \sinh^2\theta]^{1/2}} = \cosh\theta$$

and

$$\frac{dx}{d\tau} = \frac{dx}{[(dt)^2 - (dx)^2]^{1/2}} = \frac{dx/dt}{[1 - (dx/dt)^2]^{1/2}} = \frac{\beta}{[1-\beta^2]^{1/2}} = \frac{\tanh\theta}{[1-\tanh^2\theta]^{1/2}}$$

$$= \frac{\tanh\theta}{\left[\frac{\cosh^2\theta}{\cosh^2\theta} - \frac{\sinh^2\theta}{\sinh^2\theta}\right]^{1/2}} = \frac{\tanh\theta\ \cosh\theta}{[\cosh^2\theta - \sinh^2\theta]^{1/2}} = \sinh\theta$$

(In the special case chosen here the total space component of displacement dr is equal to the x displacement dx. More generally the space part of displacement dr has the value $[(dx)^2 + (dy)^2 + (dz)^2]^{1/2}$. Then it is the total space part of the unit tangent vector which has the value

$$\frac{dr}{d\tau} = \frac{\beta}{(1-\beta^2)^{1/2}} = \sinh\theta).$$

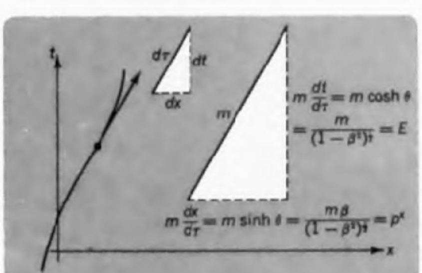


Fig. 89. The energy-momentum 4-vector. Obtained by multiplying the unit tangent vector of Fig. 88 by the constant mass m of the particle. The time component is called the "relativistic energy" and is given the label E.

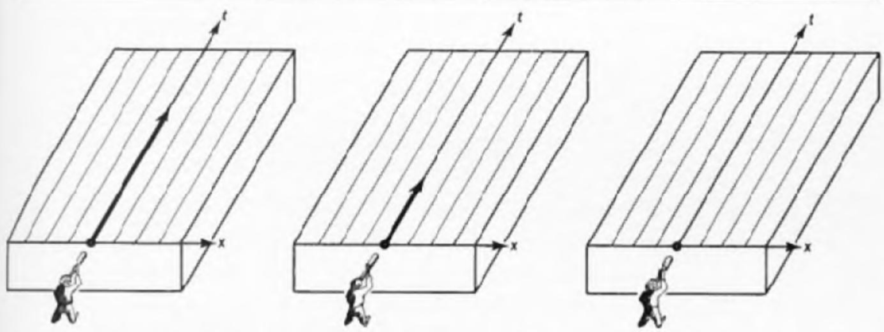
The details of this chain of ideas and alternative expressions for the space and time components of all three 4-vectors appear in the figures. No one can question that the 4-vector (dt, dx, dy, dz) remains a 4-vector after division by a factor $d\tau$ and multiplication by a factor m, both of which are the same in all frames of reference.

E-conservation in one frame follows from momentum conservation in all frames

So much for a quick introduction to the relation between momentum and energy. Now for an important question. Why can one call the time component of the preceding 4-vector by the name *energy*? For two reasons: First, because this time component has the correct units—the units of mass. Second, and more important, because the total time component is conserved in all collisions. Proof that the sum of the E values for all particles is conserved in a collision rests on a simple principle: *When three components of a 4-vector are conserved in all frames, the fourth component is also conserved* (Table 9). We know that the three (space) components of the total momentum of a system are conserved

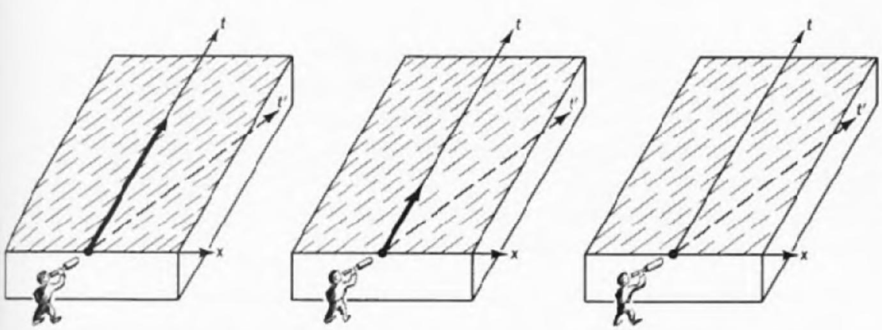
Table 9. Zero momentum change in two frames guarantees zero energy change in all frames.

RELEVANCE TO THE DISCUSSION OF MOMENTUM AND ENERGY



The vanishing of the x component of a vector in *one* frame of reference is of no help at all in drawing any conclusion about the t component of that vector. Here are three vectors of different magnitudes—one of them even of zero magnitude—all of which look alike to someone who is aware only of the x component.

The law of conservation of momentum states that the total of the momenta after the collision is equal to the total of the momenta before that collision. Equivalently, there is a certain quantity — the change in the total momentum in the collision—*which is known to be zero. This is only part of the story. We want to have full information on a complete* 4-vector *(which is equal to the* change *in the total* energy-momentum *4-vector in the collision). Looking at the space component alone (or in the diagram, verifying only that the x component of this 4-vector vanishes) does not help to show that the time component vanishes (that is, that the energy change is zero).*



A look at the same vector from another frame of reference discloses at once the difference between vectors, which all looked alike in the previous frame of reference. Let it be known that the space component of a 4-vector vanishes in two different reference frames. Then one is assured that the 4-vector itelf is truly zero (right-hand case).

The vanishing of the space component ("momentum component") of a certain 4-vector (this 4-vector being the change in the total energy-momentum *4-vector in the collision) in two different reference frames is enough to guarantee that this 4-vector itself altogether vanishes. Thus, from the fact that* momentum *is conserved in laboratory and rocket frames one concludes that* energy *is conserved in* all *frames.*

in all frames. Therefore the total time-component is also conserved. Details of this demonstration follow.

The Lorentz equations for transforming the elements of a displacement from laboratory to rocket frame can be written (Eqs. 37)

$$\begin{aligned} dt' &= -dx \sinh \theta_r + dt \cosh \theta_r \\ dx' &= dx \cosh \theta_r - dt \sinh \theta_r \\ dy' &= dy \\ dz' &= dz \end{aligned}$$

These equations remain valid when both sides are multiplied by the invariant mass m and divided by the invariant interval $d\tau = d\tau'$:

$$\begin{aligned} m\frac{dt'}{d\tau} &= -m\frac{dx}{d\tau} \sinh \theta_r + m\frac{dt}{d\tau} \cosh \theta_r \\ m\frac{dx'}{d\tau} &= m\frac{dx}{d\tau} \cosh \theta_r - m\frac{dt}{d\tau} \sinh \theta_r \\ m\frac{dy'}{d\tau} &= m\frac{dy}{d\tau} \\ m\frac{dz'}{d\tau} &= m\frac{dz}{d\tau} \end{aligned}$$

But $m\dfrac{dx}{d\tau}$, $m\dfrac{dy}{d\tau}$, and $m\dfrac{dz}{d\tau}$ are the components of the relativistic momentum and $m\dfrac{dt}{d\tau}$—the time component of the new 4-vector—is what we have chosen to call the "relativistic energy E". Thus we arrive at the following important equations connecting momentum and the new quantity "E" in one frame with momentum and E' in another inertial frame.

Lorentz transformation of momentum and energy

$$\begin{aligned} E' &= -p^x \sinh \theta_r + E \cosh \theta_r \\ p'^x &= p^x \cosh \theta_r - E \sinh \theta_r \\ p'^y &= p^y \\ p'^z &= p^z \end{aligned} \tag{78}$$

("Lorentz transformation of momentum and energy")

Now let two particles collide: $p_1{}^x$ and $p_2{}^x$ are the x components of momentum of these particles respectively before the collision as measured in the *laboratory* frame, while E_1 and E_2 are the "relativistic energies" in this frame. Similarly $p_1{}'^x$ and $p_2{}'^x$ are the x components of momentum of these particles before the collision as measured in the *rocket* frame. The two sides of the second equation (of Eqs. 78) for each particle can be added to give the total x momentum in the rocket frame before the collision.

$$(p_1{}'^x + p_2{}'^x) = (p_1{}^x + p_2{}^x) \cosh \theta_r - (E_1 + E_2) \sinh \theta_r$$

The same equation can be written for the particles that rebound from the collision (two particles for an elastic collision; one particle if amalgamation occurs; many particles if fragmentation takes place). These *before* and *after* equations can be analyzed as follows:

$$(79)\qquad \begin{pmatrix}\textit{before collision:}\\ \text{total } x \text{ momentum}\\ \text{as observed in}\\ \text{rocket frame}\end{pmatrix} = \begin{pmatrix}\textit{before collision}\\ \text{total } x \text{ momentum}\\ \text{as observed in}\\ \text{laboratory frame}\end{pmatrix}\cosh\theta_r - \begin{pmatrix}\textit{before collision:}\\ \text{total relativistic}\\ \text{energy as observed}\\ \text{in laboratory frame}\end{pmatrix}\sinh\theta_r$$

STEP 1: these terms are equal—conservation of momentum!

STEP 2: these terms are equal—conservation of momentum!

CONCLUSION: these terms are equal—proves the *conservation of relativistic energy*!

$$(80)\qquad \begin{pmatrix}\textit{after collision:}\\ \text{total } x \text{ momentum}\\ \text{as observed in}\\ \text{rocket frame}\end{pmatrix} = \begin{pmatrix}\textit{after collision:}\\ \text{total } x \text{ momentum}\\ \text{as observed in}\\ \text{laboratory frame}\end{pmatrix}\cosh\theta_r - \begin{pmatrix}\textit{after collision:}\\ \text{total relativistic}\\ \text{energy as observed}\\ \text{in laboratory frame}\end{pmatrix}\sinh\theta_r$$

Now—for the second time in this chapter—we *demand* that momentum be conserved in the collision as observed in both laboratory and rocket frames. With this requirement, each of the momentum brackets in Eq. 79 is equal to the corresponding momentum bracket in Eq. 80. If both equations are to hold true, with corresponding momentum brackets being equal, then the energy brackets must be equal also. Therefore the total relativistic energy in the laboratory frame is the same after the collision as before the collision: *total relativistic energy is conserved in the collision.*

Features of total relativistic energy

We draw three conclusions from this analysis. First, we can assign to any particle of mass m a "relativistic energy,"

$$E = m(dt/d\tau)$$

Second, when there are several particles in free motion, the relativistic energy of the system is the sum of the relativistic energies of the separate particles. Third, when these particles collide and separate, with changes in the individual energies of the several particles, the total relativistic energy of the system is the same after the collision as it was before (conservation of relativistic energy).

The additive relation between the energy of a collection of free particles and the energies of the individual particles traces back to the additive connection between total momentum and the momenta of these particles individually. This additivity indicates that the ability to calculate the energy of single particles is sufficient to allow evaluation of the energy of a collection of such particles.

Alternative expressions for energy

The expression for the relativistic energy of one particle can be written in a variety of ways, the circumstances determining which form is most useful. Thus we have (Fig. 89)

$$(81)\qquad E = m(dt/d\tau) = m/(1-\beta^2)^{1/2} = m\cosh\theta$$

What can be learned from these expressions about the relation between relativistic energy, E, and velocity? Between E and energy defined by the Newtonian expression? Between E and momentum?

For very small β, expand the expression for relativistic energy in a "power series" in powers of β using the binomial theorem or some other method:

$$E = m/(1-\beta^2)^{1/2} = m(1-\beta^2)^{-1/2} = m\left(1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 + \cdots\right)$$

For sufficiently small velocity β this series can be approximated to any desired accuracy by the first two terms

$$E \approx m(1 + \beta^2/2) = m + 1/2\, m\beta^2 \qquad \text{(low velocity)} \tag{82}$$

But $1/2\, m\beta^2$ is simply the Newtonian expression for kinetic energy in units of mass. Thus the relativistic energy E is *related* to the kinetic energy of the particle. But E is *not equal* to the kinetic energy of the particle, because of the presence of the extra term m. This extra term remains, even for a particle at rest—a particle that has no kinetic energy at all. For this reason the term m is called the *rest energy* E_{rest} *of the particle.*

$$E_{\text{rest}} = m \qquad \text{(rest energy—units of mass)} \tag{83}$$

The rest energy of a particle in conventional units $E_{\text{rest con}}$ is obtained by multiplying the rest energy in units of mass by the conversion factor c^2. This gives the famous expression

$$E_{\text{rest con}} = mc^2 \qquad \text{(rest energy—conventional units)} \tag{84}$$

Table 10. Energy which must be imparted to a hydrogen atom ($m = 1.67 \times 10^{-27}$ kilogram) to give it a speed close to the speed of light.

β	*Distance traveled from the starting line by a light flash, in a race with the particle, at the time the particle is 1 centimeter behind*	$\frac{E_{\text{con}}}{mc^2}$	$\frac{T_{\text{con}}}{mc^2}$	T_{con} *(joules)*	*An everyday equivalent of this energy*
0.5	2 centimeters	1.15	0.15	2×10^{-11}	—
0.99	1 meter	7.1	6.1	10^{-9}	—
0.99999	1 kilometer	222	221	3×10^{-8}	Kinetic energy of 1 grain of table salt dropped from a height of 1 centimeter
0.999 . . . 99 (13 nines)	10^{11} meters†	2.2×10^{6}	$\sim 2.2 \times 10^{6}$	3×10^{-4}	Kinetic energy of 1 buck-shot pellet dropped from a height of 1 centimeter
0.9999 . . . 99 (18 nines)	10^{16} meters‡	7.1×10^{8}	$\sim 7.1 \times 10^{8}$	10^{-1}	Kinetic energy of 1 buck-shot pellet dropped from a second-story window
0.9999 . . . 999 (28 nines)	10^{26} meters§	7.1×10^{13}	7.1×10^{13}	10^{4}	Kinetic energy of a motor-cycle traveling 25 miles per hour

†About two-thirds of the distance to the sun.

‡About one light year.

§Approximate distance to the most remote galaxy photographed to date.

Allowance for rest energy essential in upholding conservation law

It is impossible to have the laws of conservation of momentum and energy upheld in all inertial frames of reference unless the rest energy is included in the bookkeeping of the energy in every frame. This is the new lesson of space-time physics that never was apparent in Newtonian physics. Newtonian mechanics does not contain an expression for the rest energy of a particle. However, in Newtonian mechanics any constant energy can be added to the energy of a particle without changing the laws which describe its motion. One may think of the low-velocity limit of the relativistic expression for energy as providing this previously arbitrary constant.

Relativistic expression for kinetic energy

The relativistic energy of a particle in any reference frame can be thought of as made up of two parts: the rest energy m of the particle plus the additional energy that the particle has by virtue of its motion. This additional energy is the kinetic energy of the particle. The relativistic expression for kinetic energy is

$$\begin{aligned} (85) \qquad T &= E - E_{\text{rest}} = m \cosh \theta - m = m(\cosh \theta - 1) \\ &= m[1/(1-\beta^2)^{1/2} - 1] \end{aligned} \qquad \text{(kinetic energy—units of mass)}$$

This expression for relativistic kinetic energy is valid for all particle velocities. In contrast, *only for low particle velocities is the kinetic energy correctly given by the Newtonian formula*, $1/2\ m\beta^2$.

As the velocity increases and approaches the speed of light, the energy increases without limit. Therefore, even if one can draw upon unlimited supplies of energy, he cannot accelerate a particle to the speed of light. How rapidly the energy requirement increases as the velocity approaches the speed of light is shown by the numbers in Table 10.

Energy, the time component of the momentum-energy 4-vector and energy, the time dimension of the triangle in Fig. 90, can be evaluated by the familiar means by which one evaluates the side of any other triangle. The two principal methods employ *proportion* and the *Pythagorean theorem*. In evaluating energy as a function of velocity we made use of the similarity of the triangle $m\,E\,p$ and the triangle $d\tau\ dt\ dx$ (see Fig. 87). By proportion we found the relation $E/m = dt/d\tau = 1/(1-\beta^2)^{1/2}$. Now we want to find energy as a function of momentum. For this purpose it is enough to refer to the triangle $m\,E\,p$ all by itself. However, it is necessary to recognize that the geometry is not Euclidean, but Lorentzian. The square of the hypotenuse is given, not by the sum of the squares of the other two sides, but by their difference; thus

Mass as magnitude of momentum-energy 4-vector

$$(86) \qquad m^2 = E^2 - p^2 \qquad \text{(units of mass)}$$

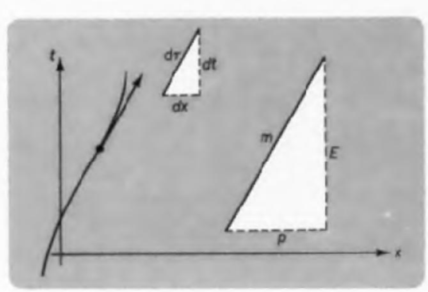


Fig. 90. The energy-momentum 4-vector.

This expression gives the squared magnitude of the momentum-energy 4-vector.† This formula is the exact analog of the formula that gives the squared magnitude of the spacetime interval between two nearby events on the world line of the particle

$$(d\tau)^2 = (dt)^2 - (dr)^2$$

In both formulas the individual quantities on the right-hand side depend upon the state of motion of the particle or the reference frame in which it is observed. In other words, the individual components of the momentum-energy 4-vector—or the energy E of the particle and its momentum p—do not have the same values in the laboratory frame as in the rocket frame. In contrast, the left-hand side of each equation—the rest mass m and the interval $d\tau$—are invariants: they have the same values in all frames of reference.

An explicit expression for energy in terms of momentum is obtained from Eq. 86 by solving for E; thus

$$E = (m^2 + p^2)^{1/2} \tag{87}$$

---

†For many purposes it is convenient to have an expression for the squared magnitude of a 4-vector in terms of the four components of that vector. One has to be a little more careful about the notation for these components than one is for vectors in three-dimensional Euclidean space. The normal representation for a 4-vector in this book and in most current literature uses *upper indices*

$$p^t = E = m\,dt/d\tau, \qquad p^x = m\,dx/d\tau, \qquad p^y = m\,dy/d\tau, \qquad p^z = m\,dz/d\tau$$

There is another representation which uses *lower indices* but with the *time component reversed in sign*

$$p_t = -m\,dt/d\tau, \qquad p_x = m\,dx/d\tau, \qquad p_y = m\,dy/d\tau, \qquad p_z = m\,dz/d\tau$$

The two alternative representations, employing upper or lower labels, are also used for other 4-vectors, as, for example, the vector R from the origin of an inertial reference frame to some chosen event; thus

$$R^t = t, \qquad R^x = x, \qquad R^y = y, \qquad R^z = z$$

and

$$R_t = -t, \qquad R_x = x, \qquad R_y = y, \qquad R_z = z$$

In terms of this notation, the invariant squared interval for a spacelike separation between event and origin takes the convenient form

$$\sigma^2 = R_tR^t + R_xR^x + R_yR^y + R_zR^z = -t^2 + x^2 + y^2 + z^2$$

When the separation is timelike, the expression for the squared interval is

$$\tau^2 = -(R_tR^t + R_xR^x + R_yR^y + R_zR^z) = t^2 - x^2 - y^2 - z^2$$

The momentum-energy 4-vector is a timelike 4-vector, for a very simple reason: because two successive events on the world line of a particle are separated by a timelike interval. Therefore the squared magnitude of this 4-vector is to be calculated from a formula analogous to the equation for τ^2; thus

$$\begin{aligned}\begin{pmatrix}\text{squared}\\ \text{magnitude}\end{pmatrix} &= -(p_tp^t + p_xp^x + p_yp^y + p_zp^z)\\ &= m^2[(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2]/(d\tau)^2\\ &= m^2\end{aligned}$$

In *Euclidean* geometry, in which a vector has only space components, this distinction between upper and lower labels is unimportant, and frequently lower labels are used exclusively. However, in spacetime geometry, in which there *is* a difference in sign for the time component associated with the difference between upper and lower labels, it is important to maintain this distinction. Moreover, the components of a 4-vector with upper labels are ordinarily more convenient to work with, as these components often relate directly to the changes in individual coordinates.

This formula is as good at arbitrarily high momentum as at low momentum. It admits of simplification in the two extreme limits.

When the momentum p is small compared to m (that is, when β is very small compared to unity: "nonrelativistic limit") one can expand Eq. 87 by using the binomial theorem or some other method, finding

Energy in terms of momentum; Newtonian and extreme relativistic limits

$$\begin{aligned} E &= m(1 + p^2/m^2)^{1/2} \\ &= m + (p^2/2m) + (p^4/8m^3) \cdots \qquad \text{(low } p\text{)} \end{aligned}$$

For sufficiently small momentum p this series can be approximated to any desired accuracy by the first two terms

$$E \approx m + p^2/2m \qquad \text{(low } p\text{)} \tag{88}$$

Here the first term represents the rest energy. The second term is the Newtonian formula for the kinetic energy of a particle of momentum p.

When the momentum p is very large compared to m ("extreme relativistic limit") one can again expand the accurate formula in a power series

$$\begin{aligned} E &= p(1 + m^2/p^2)^{1/2} \\ &\approx p + (m^2/2p) + (m^4/8p^3) \cdots \qquad \text{(high } p\text{)} \end{aligned}$$

For sufficiently high momentum p the series can be approximated to any desired accuracy by the first term

$$E \approx p \qquad \text{(extreme relativistic limit)} \tag{89}$$

In this limit the rest mass has no influence on the relation between momentum and energy.

Should one be surprised that the legs E and p of the triangle of Fig. 90 both grow greater and greater while the hypotenuse m remains fixed at a value that is small compared to that of either leg? How can the hypotenuse of a right triangle possibly remain fixed in length while the legs increase in length without limit? Such a relationship of lengths of legs and length of hypotenuse is absolutely incompatible with Euclidean geometry. But the geometry is not Euclidean. In the Lorentz geometry of spacetime the square of the hypotenuse is equal to the *difference* of the squares of the two legs. Therefore there is no paradox in the combination of a hypotenuse of constant length with two legs, E and p, which can both increase without limit and ultimately become practically equal in magnitude.

There is another way to see that energy should become approximately equal to momentum when either quantity becomes very great compared to the rest mass. Quite generally, and without any approximation, we deduce from the formulas $p = m\beta/(1 - \beta^2)^{1/2}$ and $E = m/(1 - \beta^2)^{1/2}$ the result

Momentum as rate of transport of massenergy

$$p = \beta E \qquad \text{(all velocities)} \tag{90}$$

From this formula it follows that p becomes arbitrarily close to E as the velocity becomes arbitrarily close to the speed of light.

One can give an illuminating interpretation to the formula (Eq. 90). There E represents the mass-energy of the particle, and β measures the speed with which this mass-energy is moving. Therefore the product, *the momentum p, represents* *the rate of transport of mass-energy*. It is interesting that the mass-energy factor in this formula (the quantity E in Eq. 90) differs from the mass m that a Newtonian analysis might have suggested. What accounts for the transport of mass-energy is not rest mass alone, but rest mass supplemented by the mass-equivalent of the kinetic energy; or in other words, the total mass-energy, E.

The rest mass itself does not appear in the formula $p = \beta E$. Therefore we put this formula at the center of Fig. 91, and group around it the other key formulas connecting energy, momentum, and velocity. Each relationship has its own special field of usefulness, as indicated by the captions in the diagram.

Nothing has been said in our analysis of momentum and energy about the internal structure, if there is any, of the object that carries these attributes. The object can be a rocket, or a complex organic molecule, or an elementary particle, or even a photon—the elementary quantum of light. In all these examples the motion proceeds with a speed less than that of light, except, of course, for light itself. For light traveling through a vacuum, the velocity β has a value of precisely unity. In this case the formulas $E = m/(1 - \beta^2)^{1/2}$ and $p = m\beta/(1 - \beta^2)^{1/2}$ evidently lose all usefulness. However, the relation (90) becomes extraordinarily simple; thus

Any packet of energy that moves with speed of light has zero rest mass

(91) $p = E$ (for any energy traveling with the speed of light)

Moreover, the relation $m^2 = E^2 - p^2$ tells us that in this case the rest mass is zero. We conclude that *any agency which transports energy in a straight line with the speed of light is characterized by a zero rest mass.* Today only three mechanisms for transmitting energy with the speed of light are known—elec-

Fig. 91. The quantities measured in an experiment decide which relativistic formula is convenient in the analysis of the experiment.