tors, the probability of its being seen in the other detectors is zero. Matter and radiation propagation is described by wavelike behavior, but matter and radiation interact (that is, undergo creation/annihilation or detection) as particles.

5.6 Uncertainty Principle

In Section 5.4, when we discussed the superposition of waves, we learned that to localize a wave packet over a small region Δx, we had to use a large range Δk of wave numbers. For the case of two waves, we found in Equation (5.22) that $\Delta k\, \Delta x = 2\pi$. If we examine a Gaussian wave packet closely, we would find that the product $\Delta k\, \Delta x = 1/2$. The minimum value of the product $\Delta k\, \Delta x$ is obtained when Gaussian wave packets are used.

In Section 5.4 we learned that it is impossible to measure simultaneously, with no uncertainty, the precise values of k and x for the same particle. The wave number k may be rewritten as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = p\frac{2\pi}{h} = \frac{p}{\hbar} \tag{5.37}$$

and

$$\Delta k = \frac{\Delta p}{\hbar} \tag{5.38}$$

so that, in the case of the Gaussian wave packet,

$$\Delta k\, \Delta x = \frac{\Delta p}{\hbar}\Delta x = \frac{1}{2}$$

or

$$\Delta p\, \Delta x = \frac{\hbar}{2} \tag{5.39}$$

for Gaussian wave packets.

The relationship in Equation (5.39) was first presented in 1927 by the German physicist Werner Heisenberg, who won the Nobel Prize for Physics in 1932. This uncertainty applies in all three dimensions, so we should put a subscript on Δp to indicate the x direction Δp_x. Heisenberg's **uncertainty principle** can therefore be written

Heisenberg uncertainty principle for p_x and x

$$\boxed{\Delta p_x\, \Delta x \geq \frac{\hbar}{2}} \tag{5.40}$$

which establishes limits on the simultaneous knowledge of the values of p_x and x.* The limits on Δp_x and Δx represent the lowest possible limits on the uncertainties in knowing the values of p_x and x, no matter how good an experimental measurement is made. It is possible to have a greater uncertainty in the values of p_x and x, but it is not possible to know them with more precision than allowed by the uncertainty principle. The uncertainty principle does not apply to the products of Δp_z and Δx or to that of Δp_y and Δz. The value of $\Delta p_z\, \Delta x$ can be zero. Equation (5.40) is true not only for specific waves such as water or sound, but for matter waves as

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*In some representations of the uncertainty principle, the factor $\frac{1}{2}$ is absent. Our form represents the lower limit of uncertainty.

well. It is a consequence of the de Broglie wavelength of matter. If we want to know the position of a particle very accurately, then we must accept a large uncertainty in the momentum of the particle. Similarly, if we want to know the precise value of a particle's momentum, it is not possible to specify the particle's location precisely. The uncertainty principle represents another sharp digression with classical physics, where it is assumed that it is possible to specify simultaneously and precisely both the particle's position and momentum. Because of the small value of $\hbar$, the uncertainty principle becomes important only on the atomic level.

Consider a particle for which the location is known within a width of ℓ along the x axis. We then know the position of the particle to within a distance $\Delta x \leq \ell/2$. The uncertainty principle specifies that Δp is limited by

$$\Delta p \geq \frac{\hbar}{2\,\Delta x} \geq \frac{\hbar}{\ell} \tag{5.41}$$

Because $p = mv$, we have $\Delta p = m\Delta v$, and

$$\Delta v = \frac{\Delta p}{m} \geq \frac{\hbar}{m\ell} \tag{5.42}$$

These results have some interesting implications. For example, consider a particle with low energy. What is the minimum kinetic energy such a particle can have? We can use nonrelativistic equations, so we have $K = p^2/2m$. Equation (5.41) indicates there is an uncertainty in the momentum, so we can assume the minimum value of the momentum will be at least as large as its uncertainty and $p_{\text{min}} \geq \Delta p$ to find the minimum value of the kinetic energy K_{min}.

$$K_{\text{min}} = \frac{p_{\text{min}}^2}{2m} \geq \frac{(\Delta p)^2}{2m} \geq \frac{\hbar^2}{2m\ell^2} \tag{5.43}$$

Note that this equation indicates that if we are uncertain as to the exact position of a particle, for example, an electron somewhere inside an atom of diameter ℓ, the particle can't have zero kinetic energy.

AIP/Emilio Segrè Visual Archives.

Werner Heisenberg (1901–1976) was born in Germany, where he spent his entire career at various universities including Munich, Leipzig, and Berlin. He was appointed director of the Kaiser Wilhelm Institute in Berlin in 1942, the highest scientific position in Germany. After World War II Heisenberg spent much of his effort supporting research and opportunities for young physicists and speaking out against the atomic bomb.

EXAMPLE 5.8

Calculate the momentum uncertainty of (a) a tennis ball constrained to be in a fence enclosure of length 35 m surrounding the court and (b) an electron within the smallest diameter of a hydrogen atom.

Strategy We will use Equation (5.40) to find Δp_x. The position uncertainty Δx is approximately half of the enclosure.

Solution (a) If we insert the uncertainty of the location of the tennis ball, $\Delta x = (35\text{ m})/2$, into Equation (5.40), we have

$$\Delta p_x \geq \frac{1}{2}\frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34}\,\text{J}\cdot\text{s}}{2(35\text{ m})/2} = 3 \times 10^{-36}\,\text{kg}\cdot\text{m/s}$$

We will have no problem specifying the momentum of the tennis ball!

(b) The diameter of the hydrogen atom in its lowest energy state (smallest radius) is $2a_0$. We arbitrarily take the uncertainty Δx to be half the diameter or equal to the radius, $\Delta x = a_0$.

$$\Delta x = a_0 = 0.529 \times 10^{-10}\text{ m}$$

$$\Delta p_x \geq \frac{1}{2}\frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34}\,\text{J}\cdot\text{s}}{2(0.529 \times 10^{-10}\text{ m})}$$

$$= 1 \times 10^{-24}\,\text{kg}\cdot\text{m/s}$$

This may seem like a small momentum, but for an electron with a mass of about 10^{-30} kg, it corresponds to a speed of about 10^6 m/s, which is not insignificant! Note that this is comparable to the speed of the electron in the first Bohr orbit [Equation (4.31)].

EXAMPLE 5.9

Treat the hydrogen atom as a one-dimensional entity of length $2a_0$ and determine the electron's minimum kinetic energy.

Strategy We will use the uncertainty principle to determine K_{min}. Equation (5.43) gives us the minimum kinetic energy for a particle known to be located within a distance ℓ.

Solution Equation (5.43) gives

$$K_{\text{min}} = \frac{\hbar^2}{2m\ell^2} = \frac{(\hbar c)^2}{2mc^2\ell^2}$$

$$= \frac{(197\ \text{eV}\cdot\text{nm})^2}{(2)(0.511 \times 10^6\ \text{eV})(2 \times 0.0529\ \text{nm})^2} = 3.4\ \text{eV}$$

A calculation considering three dimensions would give a result about twice this value. This simple calculation gives a reasonable value for the kinetic energy of the ground state electron of the hydrogen atom.

Energy-Time Uncertainty Principle Equation (5.40) is not the only form of the uncertainty principle. We can find another form by using Equation (5.23) from our study of wave motion. When we superimposed two waves to form a wave packet we found $\Delta\omega\ \Delta t = 2\pi$. If we evaluate this same product using Gaussian packets, we will find

$$\Delta\omega\ \Delta t = \frac{1}{2} \tag{5.44}$$

just as we did for the product $\Delta k\ \Delta x$. A relationship like this is easy to understand. If we are to localize a wave packet in a small time Δt (instead of over an infinite time as for a single wave), we must include the frequencies of many waves to have them cancel everywhere but over the time interval Δt. Because $E = hf$, we have for each wave

$$\Delta E = h\,\Delta f = h\frac{\Delta\omega}{2\pi} = \hbar\,\Delta\omega$$

Therefore

$$\Delta\omega = \frac{\Delta E}{\hbar} \quad \text{and} \quad \Delta\omega\,\Delta t = \frac{\Delta E}{\hbar}\Delta t = \frac{1}{2}$$

We can therefore obtain another form of Heisenberg's uncertainty principle:

Heisenberg uncertainty principle for energy and time

$$\boxed{\Delta E\,\Delta t \geq \frac{\hbar}{2}} \tag{5.45}$$

Other *conjugate variables* similar to p_x and x in Equation (5.40) also form uncertainty principle relations. The product of conjugate variables (such as p_x and x or E and t) must have the same dimensions as Planck's constant. Conjugate variable pairs include the angular momentum L and angle θ, as well as the rotational inertia I and angular velocity ω. Similar uncertainty relations can be written for them.

We once again must emphasize that the uncertainties expressed in Equations (5.40) and (5.45) are intrinsic. They are not due to our inability to construct better measuring equipment. No matter how well we can measure, no matter how accurate an instrument we build, and no matter how long we measure, we can never do any better than the uncertainty principle allows. Many people, including Einstein, have tried to think of situations in which it is violated, but they have not succeeded. At the 1927 Solvay conference Bohr and Einstein had several discussions about the uncertainty principle. Every morning at breakfast Einstein would present a new *gedanken* experiment that would challenge the uncertainty principle. In his careful, deliberate manner, Bohr would refute each objection. Eventually Einstein conceded—he could not provide a valid example of contradiction. These discussions continued off and on into the 1930s, because Einstein had difficulty accepting the idea that the quantum theory could give a complete description of physical phenomena. He believed that quantum theory could give a statistical description of a collection of particles but could not describe the motion of a single particle. Einstein presented several paradoxes to support his ideas. Bohr was able to analyze each paradox and present a reasonable answer. Bohr stressed his complementarity principle, which precludes a simultaneous explanation in terms of waves and particles, as well as Heisenberg's uncertainty principle.

Bohr and Einstein discussions

Let's return to the previous discussion of determining which slit an electron passes through in the double-slit experiment (see Figure 5.21). We again shine light on the electrons passing through the slits and look with a powerful microscope. This time we will use the uncertainty principle and make a more detailed calculation. Photons from the shining light bounce off the electron as the electron passes through one of the slits. Photons then scatter into the microscope where we observe them. We must be able to locate the electron's position in y to at least within $\Delta y < d/2$ (where d is the distance between the two slits) to know which slit each electron went through. If the position of the electron is uncertain to less than $d/2$, then according to the uncertainty principle, the electron's momentum must be uncertain to at least $\Delta p_y > \hbar/d$. Just by scattering photons off the electrons to know which slit the electron went through, we introduce an uncertainty in the electron's momentum. This uncertainty has been caused by the measurement itself.

Consider an electron originally moving in a particular direction; let us choose $\theta = 0$ for convenience. By scattering the photon from the electron we now have an uncertainty in the angle θ due to the "kick" given the electron by the photon in the measurement process. The uncertainty in the electron's angle due to a possible momentum change along the y axis is $\Delta\theta = \Delta p_y/p$, but because $p = h/\lambda$, we have

$$\Delta\theta = \frac{\Delta p_y}{p} = \frac{(\Delta p_y)\lambda}{h} = \frac{(\hbar)\lambda}{hd} = \frac{\lambda}{2\pi d}$$

According to Equation (5.36) the first interference maximum will be at $\sin\theta = \lambda/d$ and the first minimum at $\sin\theta = \lambda/2d$. For small angle scattering, $\sin\theta \approx \theta$, and the angle of the first minimum is $\theta_{\text{min}} \approx \lambda/2d$. Note that the position of the first minimum is on the same order as our uncertainty in $\Delta\theta$, so the interference pattern is washed out. If we insist on identifying the electrons as particles and knowing which slit the electrons pass through, the wave characteristics of the electron disappear. We cannot simultaneously treat the electron as both a particle and a wave. This limitation seems to be a fundamental characteristic of the laws of nature. Only the smallness of Planck's constant h keeps us from encountering this limitation in everyday life.

Niels Bohr tried to turn this limitation into a philosophical principle. When he was awarded the Danish Order of the Elephant, he featured on his coat of arms (see Figure 5.22) the Chinese yin-yang symbol, which stands for the two

AIP/Niels Bohr Library, Margarethe Bohr Collection.

Figure 5.22 Niels Bohr's coat of arms was designed in 1947 when he was awarded the Danish Order of the Elephant. This award was normally given only to royalty and foreign presidents. Bohr chose the Chinese yin-yang symbol because it stands for the two opposing but inseparable elements of nature. The translation of the Latin motto is "Opposites are complementary." It was hung near the king's coat of arms in the church of Frederiksborg Castle at Hillerod.

opposing but inseparable elements in nature. The Latin motto on the center of the coat of arms means “Opposites are complementary.”

EXAMPLE 5.10

Calculate the minimum kinetic energy of an electron that is localized within a typical nuclear radius of 6×10^{-15} m.

Strategy Let’s assume the minimum electron momentum is equal to that determined by the uncertainty principle for an electron constrained within the distance Δx equal to the nuclear radius ($\Delta x = \pm r$). We can then determine the minimum electron energy from the minimum momentum.

Solution Given that $\Delta x \approx r = 6 \times 10^{-15}$ m, we have

$$\begin{aligned} \Delta p \geq \frac{\hbar}{2\,\Delta x} &= \frac{6.58 \times 10^{-16}\ \mathrm{eV \cdot s}}{1.2 \times 10^{-14}\ \mathrm{m}} \\ &\geq (5.48 \times 10^{-2}\ \mathrm{eV \cdot s/m})\left(\frac{3 \times 10^{8}\ \mathrm{m/s}}{c}\right) \\ &\geq 1.64 \times 10^{7}\ \mathrm{eV}/c \end{aligned}$$

Because we assumed that the momentum p is at least as large as the uncertainty in p, we have

$$p \approx \Delta p \geq 1.64 \times 10^{7}\ \mathrm{eV}/c$$

Because we don’t yet know the electron’s energy, let’s be careful and calculate it relativistically.

$$\begin{aligned} E^2 &= (pc)^2 + E_0^2 \\ &= \left[\left(1.64 \times 10^{7}\ \frac{\mathrm{eV}}{c}\right)c\right]^2 + (0.511\ \mathrm{MeV})^2 \\ &= (16.4\ \mathrm{MeV})^2 + (0.511\ \mathrm{MeV})^2 \\ E &= 16.4\ \mathrm{MeV} \\ \mathrm{K.E.} &= E - E_0 = 16.4\ \mathrm{MeV} - 0.51\ \mathrm{MeV} \\ &= 15.9\ \mathrm{MeV} \end{aligned}$$

Note that because $E \gg E_0$, a relativistic calculation was needed.

CONCEPTUAL EXAMPLE 5.11

We found in the last example that if an electron is confined within the size of a nuclear radius, the uncertainty principle suggests that the minimum kinetic energy of the electron must have a minimum value of about 16 MeV. What does this indicate about the possibility of electrons existing within the nucleus?

Solution The value of 16 MeV for the electron’s kinetic energy is larger than that observed for electrons emitted from nuclei in beta decay. We conclude that electrons are not confined within the nucleus. Electrons emitted from the nucleus (during beta decay) must actually be created when they are emitted.

EXAMPLE 5.12

An atom in an excited state normally remains in that state for a very short time ($\sim 10^{-8}$ s) before emitting a photon and returning to a lower energy state. The “lifetime” of the excited state can be regarded as an uncertainty in the time Δt associated with a measurement of the energy of the state. This, in turn, implies an “energy width,” namely, the corresponding energy uncertainty ΔE. Calculate (a) the characteristic “energy width” of such a state and (b) the uncertainty ratio of the frequency $\Delta f/f$ if the wavelength of the emitted photon is 300 nm.

Strategy (a) We use the uncertainty principle, Equation (5.45), to determine ΔE because we know Δt.

(b) We can determine Δf from the energy uncertainty ΔE by using $E = hf$: $\Delta E = h\,\Delta f$. We can determine the frequency by $f = c/\lambda$.

Solution (a) Equation (5.45) gives

$$\Delta E \geq \frac{\hbar}{2\,\Delta t} = \frac{6.58 \times 10^{-16}\ \text{eV}\cdot\text{s}}{(2)(10^{-8}\ \text{s})} = 3.3 \times 10^{-8}\ \text{eV}$$

This is a small energy, but many excited energy states have such energy widths. For stable ground states, $\tau = \infty$, and $\Delta E = 0$. For excited states in the nucleus, the lifetimes can be as short as 10^{-20} s (or shorter) with energy widths of 100 keV (or more).

(b) The frequency is found to be

$$f = \frac{c}{\lambda} = \frac{3 \times 10^{8}\ \text{m/s}}{300 \times 10^{-9}\ \text{m}} = 10^{15}\ \text{Hz} \tag{5.46}$$

The uncertainty Δf is

$$\Delta f = \frac{\Delta E}{h} = \frac{3.3 \times 10^{-8}\ \text{eV}}{4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}} = 8 \times 10^{6}\ \text{Hz} \tag{5.47}$$

The uncertainty ratio of the frequency $\Delta f/f$ is

$$\frac{\Delta f}{f} = \frac{8 \times 10^{6}\ \text{Hz}}{10^{15}\ \text{Hz}} = 8 \times 10^{-9}$$

Modern instruments are capable of measuring ratios approaching 10^{-17}, or 1 Hz in a frequency of 10^{17} Hz! Experimental physicists have managed to improve this ratio by an irregular factor of 100 every three years over the past two decades. The experimental limitations are considerably better than needed to measure the energy widths.

5.7 Probability, Wave Functions, and the Copenhagen Interpretation

We learned in elementary physics that the instantaneous wave intensity of electromagnetic radiation (light) is $\epsilon_0 c E^2$ where E is the electric field. Thus the probability of observing light is proportional to the square of the electric field. In the double-slit light experiment we can be assured that the electric field of the light wave is relatively large at the bright spots on the screen and small in the region of the dark places.

If Young's double-slit experiment is performed with very low intensity levels of light, individual flashes can be seen on the observing screen. We show a simulation of the experiment in Figure 5.19. After only 20 flashes (Figure 5.19a) we cannot make any prediction as to the eventual pattern, but we still know that the *probability* of observing a flash is proportional to the square of the electric field. We now briefly review this calculation that is normally given in introductory physics courses. If the distance from the central ray along the screen we are observing in an experiment like that depicted in Figure 5.18a is denoted by y, the probability for the photon to be found between y and $y + dy$ is proportional to the intensity of the wave (E^2) times dy. For Young's double-slit experiment, the value of the electric field $\vec{E}$ produced by the two interfering waves is large where the flash is likely to be observed and small where it is not likely to be seen. By counting the number of flashes we relate the energy flux I (called the intensity) of the light to the number flux, N per unit area per unit time, of photons having energy hf. In the wave description, we have $I = \epsilon_0 c\,\langle E^2\rangle$, and in what appears to be the particle description, $I = Nhf$. The flux of photons N, or the probability P of observing the photons, is proportional to the average value of the square of the electric field $\langle E^2\rangle$.

How can we interpret the probability of finding the electron in the wave description?

First, let's remember that the localization of a wave can be accomplished by using a wave packet. We used a function $\Psi(x, t)$ to denote the superposition of many waves to describe the wave packet. We call this function $\Psi(x, t)$ the **wave function**. In the case of light, we know that the electric field $\vec{E}$ and magnetic field $\vec{B}$ satisfy a wave equation. In electrodynamics either $\vec{E}$ or $\vec{B}$ serves as the wave function Ψ. For particles (say electrons) a similar behavior occurs. In this case the wave function $\Psi(x, t)$ determines the probability, just as the flux of photons N arriving at the screen and the electric field $\vec{E}$ determined the probability in the case of light.

Wave function

For matter waves having a de Broglie wavelength, it is the wave function Ψ that determines the likelihood (or probability) of finding a particle at a particular position in space at a given time. The value of the wave function Ψ has no physical significance itself, and as we will see later, it can have a **complex** value (containing both real and imaginary numbers). The quantity $|\Psi|^2$ is called the **probability density** and represents the probability of finding the particle in a given unit volume at a given instant of time.

Probability density

In general, $\Psi(x, y, z, t)$ is a complex quantity and depends on the spatial coordinates x, y, and z as well as time t. The complex nature will be of no concern to us: we use Ψ times its complex conjugate Ψ^* when finding probabilities. We are interested here in only a single dimension y along the observing screen and for a given time t. In this case $\Psi^*\Psi\ dy = |\Psi|^2\ dy$ is the probability of observing an electron in the interval between y and $y + dy$ at a given time, and we call this $P(y)\ dy$.

$$P(y)\ dy = |\Psi(y, t)|^2\ dy \tag{5.48}$$

Because the electron has to have a probability of unity of being observed *somewhere* along the screen, we integrate the probability density over all space by integrating over y from $-\infty$ to ∞. This process is called **normalization**.

Normalization

$$\int_{-\infty}^{\infty} P(y)\ dy = \int_{-\infty}^{\infty} |\Psi(y, t)|^2\ dy = 1 \tag{5.49}$$

Max Born (Nobel Prize, 1954), one of the founders of the quantum theory, first proposed this probability interpretation of the wave function in 1926. The determination of the wave function $\Psi(x, t)$ is discussed in much more detail in the next chapter.

The use of wave functions $\Psi(x, y, z, t)$ rather than the classical positions $x(t)$, $y(t)$, $z(t)$ represents a clean break between classical and modern physics. Physicists have developed a set of rules and procedures in quantum theory to determine physical observables like position, momentum, and energy (see Section 6.2).

The Copenhagen Interpretation

Erwin Schrödinger and Werner Heisenberg worked out independent and separate mathematical models for the quantum theory in 1926. We examine Schrödinger's theory in Chapter 6, because it is somewhat easier to understand and is based on waves. Paul Dirac reported his relativistic quantum theory in 1928. Today there is little disagreement about the mathematical formalism of quantum theory. That is not the case regarding its interpretation.

We want to examine the *Copenhagen interpretation*, because it is the mainstream interpretation of quantum theory. Werner Heisenberg announced his uncertainty principle in early 1927 while he was a lecturer in Bohr's Institute of Theoretical Physics. At first Bohr, the mentor, thought Heisenberg's uncertainty principle was too narrow, and he pointed out a mistake in Heisenberg's paper concerning a *gedanken* experiment about a gamma-ray microscope used by Heisenberg to prove his point. Heisenberg, the 25-year-old rising star, strongly objected at first to Bohr's opinion and refused Bohr's suggestion to withdraw his paper on the uncertainty principle. Bohr and Heisenberg had many discussions in 1927 formulating the interpretation of quantum mechanics now known as the "Copenhagen interpretation," "Copenhagen school," or sometimes unkindly as "Copenhagen orthodoxy." It was strongly supported by Max Born and Wolfgang Pauli (profiled in Chapter 8).

There are various formulations of the interpretation, but it is generally based on the following:

1. The uncertainty principle of Heisenberg
2. The complementarity principle of Bohr
3. The statistical interpretation of Born, based on probabilities determined by the wave function

Together these three concepts form a logical interpretation of the physical meaning of quantum theory. According to the Copenhagen interpretation, physics depends on the outcomes of measurement. Consider a single electron passing through the two-slit experiment. We can determine precisely where the electron hits the screen by noting a flash. The Copenhagen interpretation rejects arguments about where the electron was between the times it was emitted in the apparatus (and subsequently passed through the two slits) and when it flashed on the screen. The measurement process itself randomly chooses one of the many possibilities allowed by the wave function, and the wave function instantaneously changes to represent the final outcome. Bohr argued that it is not the task of physics to find out how nature is, because we can never understand the quantum world or assign physical meaning to the wave function. Bohr and Heisenberg argued that measurement outcomes are the only reality in physics.

Many physicists objected (and some still do!) to the Copenhagen interpretation for widely varying reasons. One of the basic objections is to its nondeterministic nature. Some also object to the vague measurement process that converts probability functions into nonprobabilistic measurements. Famous physicists who objected to the Copenhagen interpretation were Albert Einstein, Max Planck, Louis de Broglie, and Erwin Schrödinger. Einstein and Schrödinger never accepted the Copenhagen interpretation. Einstein was particularly bothered by the reliance on probabilities, and he wrote Born in 1926 that "God does not throw dice." Nonetheless, it is fair to say that the great majority of physicists today accept the Copenhagen interpretation as the primary interpretation of quantum mechanics. In the past decade physicists have used feedback systems to demonstrate that quantum indeterminism can be reduced by guiding the outcome of a probabilistic quantum process toward a deterministic outcome.*

Several paradoxes have been proposed by physicists to refute the Copenhagen interpretation. They include the famous Schrödinger cat paradox,† the

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Max Born (1882–1970) was born a German in what is now Poland. After studying at several European universities he received his degree in 1907 from the University of Göttingen. After visiting several universities and serving in World War I, he became a professor at Göttingen in 1921 where he did his most important work on the statistical meaning of the new quantum theory (Nobel Prize in Physics, 1954). He and his student, Werner Heisenberg, collaborated on the matrix mechanics version of quantum mechanics. Born, a Jew, was forced to emigrate from Germany in 1933, and after visiting Italy, Cambridge, and India, he settled at the University of Edinburgh in 1936, from which he retired in 1953.

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*See, for example, J. M. Geremia, J. K. Stockton, and H. Mabuchi, *Science* **304**, 270 (2004).

†Schrödinger published an essay in 1935, "The Present Situation in Quantum Mechanics," in which he described a thought experiment where a cat in a closed box either lived or died according to whether a quantum event occurred. The paradox was that it was not possible to know whether the cat was dead or alive until an observer opened the box, an apparent contradiction to the intuitive notion that the cat is either alive or dead at any moment.

Einstein-Podolsky-Rosen paradox,* and Bell's theorem (or inequality).† Space does not allow us to describe these paradoxes (see Problems 49-51). A Princeton University graduate student, Hugh Everett III, announced an alternate interpretation to the Copenhagen view in 1957. In Everett's "Many Worlds" interpretation, the concept of parallel universes is invoked—in itself such a weird idea that it has not gained wide acceptance, but it overcomes some objections to the Copenhagen interpretation. Since 1957 there have been several versions of the Many Worlds interpretation presented, and some physicists prefer it over the Copenhagen interpretation. Nevertheless, the Copenhagen interpretation remains the favored interpretation.

5.8 Particle in a Box

Let's now consider the situation of a particle of mass m trapped in a one-dimensional box of width ℓ. We have already used the uncertainty principle in Equation (5.43) to calculate the minimum kinetic energy of such a particle. Now let's determine the possible energies of such a particle. Because of our discussion in the previous section we want to use the wave nature of the particle in this determination.

First, what is the most probable location of the particle in the state with the lowest energy at a given time, say $t = 0$, so that $\Psi(x, 0) = \Psi(x)$? To find the probable location, we will treat the particle as a sinusoidal wave. The particle cannot be physically outside the confines of the box, so the amplitude of the wave motion must vanish at the walls and beyond. In the language of the wave function, its probability of being outside is zero, so the wave function must vanish outside. The wave function must be continuous, and the probability distribution can have only one value at each point in the box. For the probability to vanish at the walls, we must have an integral number of half wavelengths $\lambda/2$ fit into the box. Note that all the possible waves shown in Figure 5.23 fit this requirement.

The requirement of an integral number of half wavelengths $\lambda/2$ means that

$$\frac{n\lambda}{2} = \ell \quad \text{or} \quad \lambda_n = \frac{2\ell}{n} \quad (n = 1, 2, 3, \ldots) \tag{5.50}$$

The possible wavelengths are quantized, and the wave shapes will have $\sin(n\pi x/\ell)$ factors. If we treat the problem nonrelativistically and assume there is no potential energy, the energy E of the particle is

$$E = \text{K.E.} = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$

If we insert the values for λ_n, we have

$$E_n = \frac{h^2}{2m}\left(\frac{n}{2\ell}\right)^2 = n^2\frac{h^2}{8m\ell^2} \quad (n = 1, 2, 3, \ldots) \tag{5.51}$$

Therefore, the possible energies of the particle are quantized, and each of these energies E_n is a possible energy level. Note that the lowest energy is $E_1 = h^2/8m\ell^2$. Because we assumed the potential energy to be zero, E_n is also equal to the kinetic energy. We previously found in Equation (5.43) a value for $K_{\text{min}} = \hbar^2/2m\ell^2$,

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*A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? *Physical Review* **47**, 777 (1935).

†J. S. Bell, On the Einstein Podolsky Rosen paradox, *Physics* **1**, 195 (1964).