Conceptual Example 4.4

How can we find the distance of closest approach between a bombarding particle and a target scatterer of like charge?

Solution We can find this distance of closest approach for a given kinetic energy K and impact parameter b. The minimum separation occurs for a head-on collision. The bombarding particle turns around and scatters backward at 180°. At the instant the particle turns around, the entire kinetic energy has been converted into Coulomb potential energy. By setting the original (maximum) kinetic energy equal to the Coulomb potential energy when $r = r_{\text{min}}$, we can then solve the resulting equation for r_{min}. Let K be the original kinetic energy of the bombarding particle.

$$K = \frac{(Z_1 e)(Z_2 e)}{4\pi\epsilon_0 r_{\text{min}}} \tag{4.14}$$

We solve this equation to determine r_{min}.

$$r_{\text{min}} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 K} \tag{4.15}$$

Example 4.5

Rutherford found deviations from his Equation (4.13) at backward angles when he scattered 7.7-MeV α particles ($Z_1 = 2$) on aluminum ($Z_2 = 13$). He suspected this was because the α particle might be affected by approaching the nucleus so closely. Estimate the size of the nucleus based on these data.

Strategy We have just determined the distance of closest approach for the α particle, which occurs for a head-on collision or scattering angle of 180°. We propose that the r_{min} in this case is close to the sum of the α particle (^{4}He nucleus) radius and the aluminum nuclear radius.

Solution We insert the values for the α particle incident on aluminum into Equation (4.15) to find r_{min}.

$$r_{\text{min}} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 K}$$

$$= \frac{(2)(13)(1.60 \times 10^{-19}\,\text{C})^2(8.99 \times 10^9\,\text{N}\cdot\text{m}^2\cdot\text{C}^{-2})}{(7.7\,\text{MeV})(1.60 \times 10^{-13}\,\text{J/MeV})}$$

$$= 4.9 \times 10^{-15}\,\text{m}$$

We find the sum of the ^{4}He and aluminum nuclear radii to be about 5×10^{-15} m.

We will see in Chapter 12 that the aluminum nuclear radius is about twice as large as that of ^{4}He, and our approximate result here is in fair agreement with modern data. We now know that nuclear radii vary from 1×10^{-15} to 8×10^{-15} m. Thus when α particles scatter from aluminum, an α particle may approach the nucleus close enough to be affected by the nuclear force (see Chapter 12).

4.3 The Classical Atomic Model

After Rutherford presented his calculations of charged-particle scattering in 1911 and the experimental verification by his group in 1913, it was generally conceded that the atom consisted of a small, massive, positively charged "nucleus" surrounded by moving electrons. Thomson's plum-pudding model was definitively excluded by the data. Actually, Thomson had previously considered a planetary model resembling the solar system (where the planets move in elliptical orbits about the sun) but had rejected it because, although both gravitational and Coulomb forces vary inversely with the square of the distance, the planets *attract* one another while orbiting around the sun, whereas the electrons would *repel* one another. Thomson considered this to be a fatal flaw from his knowledge of planetary theory.

In order to examine the failure of the planetary model, let us examine the simplest atom, hydrogen. We will assume circular electron orbits for simplicity rather than the more general elliptical ones. The force of attraction on the electron due to the nucleus (charge $= +e$) is

$$\vec{F}_e = \frac{-1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{e}_r \tag{4.16}$$

where the negative sign indicates the force is attractive and $\hat{e}_r$ is a unit vector in the direction from the nucleus to the electron. This electrostatic force provides the centripetal force needed for the electron to move in a circular orbit at constant speed. Its radial acceleration is

$$a_r = \frac{v^2}{r} \tag{4.17}$$

where v is the tangential velocity of the electron. Newton's second law now gives

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r} \tag{4.18}$$

and

$$v = \frac{e}{\sqrt{4\pi\epsilon_0 mr}} \tag{4.19}$$

where we are using m without a subscript to be the electron's mass. When it is not clear what particle m refers to, we write the electron mass as m_e.

Example 4.6

Are we justified in using a nonrelativistic treatment for the speed of an electron in the hydrogen atom?

Strategy We use Equation (4.19) to calculate the electron's speed. If it is less than 1% of the speed of light, we are justified in using a nonrelativistic treatment. One difficulty is knowing the radius of the hydrogen atom. The size of an atom was thought to be about 10^{-10} m, so we let $r = 0.5 \times 10^{-10}$ m in order to estimate the electron's velocity.

Solution Equation (4.19) gives

$$v \approx \frac{(1.6 \times 10^{-19}\ \mathrm{C})(9 \times 10^{9}\ \mathrm{N \cdot m^2/C^2})^{1/2}}{(9.11 \times 10^{-31}\ \mathrm{kg})^{1/2}(0.5 \times 10^{-10}\ \mathrm{m})^{1/2}}$$

$$\approx 2.2 \times 10^{6}\ \mathrm{m/s} < 0.01c$$

This justifies a nonrelativistic treatment.

The kinetic energy of the system is due to the electron, $K = mv^2/2$. The nucleus is so massive compared to the electron ($m_{\mathrm{proton}} = 1836m$) that the nucleus may be considered to be at rest. The potential energy V is simply $-e^2/4\pi\epsilon_0 r$, so the total mechanical energy is

$$E = K + V = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r} \tag{4.20}$$

If we substitute for v from Equation (4.19), we have

$$E = \frac{e^2}{8\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} = \frac{-e^2}{8\pi\epsilon_0 r} \tag{4.21}$$

The total energy is negative, indicating a bound system.

Thus far, the classical atomic model seems plausible. The problem arises when we consider that the electron is accelerating due to its circular motion about the nucleus. We know from classical electromagnetic theory that an accelerated electric charge continuously radiates energy in the form of electromagnetic radiation. If the electron is radiating energy, then the total energy E of the system, Equation (4.21), must decrease continuously. In order for this to happen, the radius r must decrease. The electron will continuously radiate energy as the electron orbit becomes smaller and smaller until the electron crashes into the nucleus! This process, displayed in Figure 4.14, would occur in about 10^{-9} s (see Problem 18).

Planetary model is doomed.

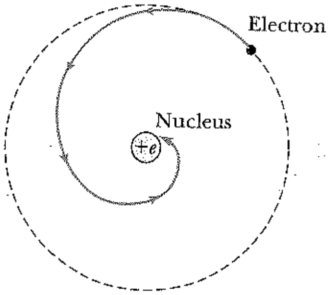

Figure 4.14 The electromagnetic radiation of an orbiting electron in the planetary model of the atom will cause the electron to spiral inward until it crashes into the nucleus.

Thus the classical theories of Newton and Maxwell, which had served Rutherford so well in his analysis of α-particle scattering and had thereby enabled him to discover the nucleus, also led to the failure of the planetary model of the atom. Physics had reached a decisive turning point like that encountered in 1900 with Planck's revolutionary hypothesis of the quantum behavior of radiation. In the early 1910s, however, the answer would not be long in coming, as we shall see in the next section.

4.4 The Bohr Model of the Hydrogen Atom

Shortly after receiving his Ph.D. from the University of Copenhagen in 1911, the 26-year-old Danish physicist Niels Bohr traveled to Cambridge University to work with J. J. Thomson and then with Rutherford at Manchester for a few months in 1912 where he became particularly involved in the mysteries of the new Rutherford model of the atom. Bohr returned to the University of Copenhagen the summer of 1912 with many questions about atomic structure. Like several others, he believed that a fundamental length about the size of an atom (10^{-10} m) was needed for an atomic model. This fundamental length might somehow be connected to Planck's new constant h. The pieces finally came together during the fall and winter of 1912 when Bohr learned of new precise measurements of the hydrogen spectrum and of the empirical formulas describing them. He set out to find a fundamental basis from which to derive the Balmer formula [Equation (3.12)], the Rydberg equation [Equation (3.13)], and Ritz's combination principles (see Problem 19).

Bohr was well acquainted with Planck's work on the quantum nature of radiation. Like Einstein, Bohr believed that quantum principles should govern more phenomena than just the blackbody spectrum. He was impressed by Einstein's application of the quantum theory to the photoelectric effect and to the specific heat of solids (see Chapter 9 for the latter) and wondered how the quantum theory might affect atomic structure.

In 1913, following several discussions with Rutherford during 1912 and 1913, Bohr published the paper* "On the Constitution of Atoms and Molecules." He

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*Niels Bohr, *Philosophical Magazine* **26**, 1 (1913) and **30**, 394 (1915).

subsequently published several other papers refining and restating his "assumptions" and their predicted results. We will generally follow Bohr's papers in our discussion.

Bohr assumed that electrons moved around a massive, positively charged nucleus. We will assume for simplicity (as did Bohr at first) that the electron orbits are circular rather than elliptical and that the nuclear mass is so much greater than the electron's mass that it may be taken to be infinite. The electron has charge $-e$ and mass m and revolves around a nucleus of charge $+e$ in a circle of radius a. The size of the nucleus is small compared to the atomic radius a.

Bohr's model may best be summarized by the following "*general assumptions*" of his 1915 paper:

Bohr's general assumptions

A. Certain "stationary states" exist in atoms, which differ from the classical stable states in that the orbiting electrons do not continuously radiate electromagnetic energy. The stationary states are states of definite total energy.

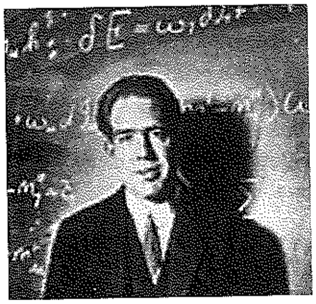
B. The emission or absorption of electromagnetic radiation can occur only in conjunction with a transition between two stationary states. The frequency of the emitted or absorbed radiation is proportional to the difference in energy of the two stationary states (1 and 2):

$$E = E_1 - E_2 = hf$$

where h is Planck's constant.

C. The dynamical equilibrium of the system in the stationary states is governed by classical laws of physics, but these laws do not apply to transitions between stationary states.

D. The mean value K of the kinetic energy of the electron-nucleus system is given by $K = nhf_{\text{orb}}/2$, where f_{orb} is the frequency of rotation. For a circular orbit, Bohr pointed out that this assumption is equivalent to the angular momentum of the system in a stationary state being an integral multiple of $h/2\pi$. (This combination of constants occurs so often that we give it a separate symbol, $\hbar \equiv h/2\pi$, pronounced "h bar.")



Niels Bohr (1885–1962) was more than just a discoverer of modern physics theories. Born in Denmark, he was the son of a university professor and began high school at about the time Planck announced his results. After his education in Denmark, Bohr traveled to England in 1911 where he worked first with J. J. Thomson and later with Ernest Rutherford. Bohr nurtured many young theoretical physicists in his Institute of Theoretical Physics (now called the Niels Bohr Institute) formed in Copenhagen in 1921, the year before Bohr won the Nobel Prize. *AIP Niels Bohr Library.*

These four assumptions were all that Bohr needed to derive the Rydberg equation. Bohr believed that Assumptions A and C were self-evident because atoms were stable: atoms exist and do not continuously radiate energy (therefore Assumption A). It also seemed that the classical laws of physics could not explain the observed behavior of the atom (therefore Assumption C).

Bohr later stated (1915) that Assumption B "appears to be necessary in order to account for experimental facts." Assumption D was the hardest for Bohr's critics to accept. It is central to the derivation of the binding energy of the hydrogen atom in terms of fundamental constants; hence Bohr restated and defended it in several ways in his papers. We have emphasized here the quantization of angular momentum aspect of Assumption D. This leads to a particularly simple derivation of the Rydberg equation.

Bohr chose his four assumptions to keep as much as possible of classical physics by introducing just those new ideas that were needed to explain experimental data. Bohr's recognition that something new was needed and his attempt to tie this to Planck's quantum hypothesis represented an advance in understanding perhaps even greater than Einstein's theory of the photoelectric effect.

Let us now proceed to derive the Rydberg equation using Bohr's assumptions. The total energy (potential plus kinetic) of a hydrogen atom was derived previously in Equation (4.21). For circular motion, the magnitude of the angular momentum L of the electron is

$$L = |\vec{r} \times \vec{p}| = mvr$$

Assumption D states this should equal $n\hbar$:

$$L = mvr = n\hbar \tag{4.22a}$$

Principal quantum number

where n is an integer called the **principal quantum number**. We solve the previous equation for the velocity and obtain

$$v = \frac{n\hbar}{mr} \tag{4.22b}$$

Equation (4.19) yields an independent relation between v and r. If we determine v^2 from Equations (4.19) and (4.22b) and set them equal, we find

$$v^2 = \frac{e^2}{4\pi\epsilon_0 mr} = \frac{n^2\hbar^2}{m^2r^2} \tag{4.23}$$

From Equation (4.23) we see that only certain values of r are allowed.

$$r_n = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2} \equiv n^2 a_0 \tag{4.24}$$

where the **Bohr radius** a_0 is given by

$$\begin{aligned} a_0 &= \frac{4\pi\epsilon_0\hbar^2}{me^2} \\ &= \frac{(1.055 \times 10^{-34}\,\mathrm{J\cdot s})^2}{\left(8.99 \times 10^9 \frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)(9.11 \times 10^{-31}\,\mathrm{kg})(1.6 \times 10^{-16}\,\mathrm{C})^2} \\ &= 0.53 \times 10^{-10}\,\mathrm{m} \end{aligned}$$

Bohr radius

Notice that the smallest diameter of the hydrogen atom is $2r_1 = 2a_0 \approx 10^{-10}$ m, the suspected (now known) size of the hydrogen atom! Bohr had found the fundamental length a_0 that he sought in terms of the fundamental constants ϵ_0, h, e, and m. This fundamental length is determined for the value $n = 1$. Note from Equation (4.24) that the atomic radius is now quantized. The quantization of various physical values arises because of the principal quantum number n. The value $n = 1$ gives the radius of the hydrogen atom in its lowest energy state (called the "ground" state). The values of $n > 1$ determine other possible radii where the hydrogen atom is in an "excited" state.

The energies of the stationary states can now be determined from Equations (4.21) and (4.24).

$$E_n = -\frac{e^2}{8\pi\epsilon_0 r_n} = -\frac{e^2}{8\pi\epsilon_0 a_0 n^2} \equiv -\frac{E_0}{n^2} \tag{4.25}$$

Quantized energy states

The lowest energy state ($n = 1$) is $E_1 = -E_0$ where

$$E_0 = \frac{e^2}{8\pi\epsilon_0 a_0} = \frac{e^2}{(8\pi\epsilon_0)}\frac{me^2}{4\pi\epsilon_0\hbar^2} = \frac{me^4}{2\hbar^2(4\pi\epsilon_0)^2} = 13.6\ \mathrm{eV} \tag{4.26}$$

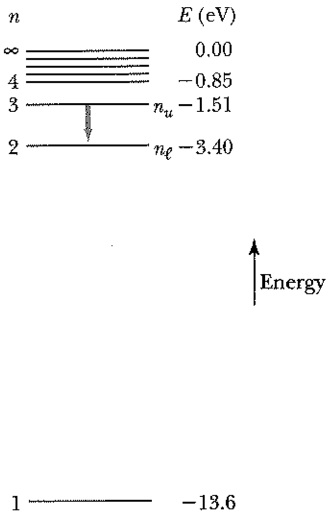


Figure 4.15 The energy-level diagram of the hydrogen atom. The principal quantum numbers n are shown on the left, with the energy of each level indicated on the right. The ground-state energy is -13.6 eV; negative total energy indicates a bound, attractive system. When an atom is in an excited state (for example, $n_u = 3$) and decays to a lower stationary state (for example, $n_\ell = 2$), the hydrogen atom must emit the energy difference in the form of electromagnetic radiation; that is, a photon emerges.

This is the experimentally measured ionization energy of the hydrogen atom. Bohr's Assumptions C and D imply that the atom can exist only in "stationary states" with definite, quantized energies E_n, displayed in the **energy-level diagram** of Figure 4.15. Emission of a quantum of light occurs when the atom is in an excited state (quantum number $n = n_u$) and decays to a lower energy state ($n = n_\ell$). A transition between two energy levels is schematically illustrated in Figure 4.15. According to Assumption B we have

$$hf = E_u - E_\ell \tag{4.27}$$

where f is the frequency of the emitted light quantum (photon). Because $\lambda f = c$, we have

$$\begin{aligned} \frac{1}{\lambda} &= \frac{f}{c} = \frac{E_u - E_\ell}{hc} \\ &= \frac{-E_0}{hc}\left(\frac{1}{n_u^2} - \frac{1}{n_\ell^2}\right) = \frac{E_0}{hc}\left(\frac{1}{n_\ell^2} - \frac{1}{n_u^2}\right) \end{aligned} \tag{4.28}$$

where

$$\frac{E_0}{hc} = \frac{me^4}{4\pi c\hbar^3(4\pi\epsilon_0)^2} \equiv R_\infty \tag{4.29}$$

This constant R_∞ is called the **Rydberg constant** (for an infinite nuclear mass). Equation (4.28) becomes

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{n_\ell^2} - \frac{1}{n_u^2}\right) \tag{4.30}$$

which is similar to the Rydberg equation (3.13). The value of $R_\infty = 1.097373 \times 10^7\ \text{m}^{-1}$ calculated from Equation (4.29) agrees well with the experimental values given in Chapter 3, and we will obtain an even more accurate result in the next section.

Bohr's model predicts the frequencies (and wavelengths) of all possible transitions in atomic hydrogen. Several of the series are shown in Figure 4.16. The Lyman series represents transitions to the lowest state with $n_\ell = 1$; the Balmer series results from downward transitions to the stationary state $n_\ell = 2$; and the Paschen series represents transitions to $n_\ell = 3$. As mentioned in Section 3.3, not

Bohr predicted new hydrogen wavelengths

all of these series were known experimentally in 1913, but it was clear that Bohr had successfully accounted for the known spectral lines of hydrogen.

The frequencies of the photons in the emission spectrum of an element are directly proportional to the differences in energy of the stationary states. When we pass white light (composed of all visible photon frequencies) through atomic hydrogen gas, we find that certain frequencies are absent. This pattern of dark

Absorption and emission spectrum

lines is called an **absorption spectrum**. The missing frequencies are *precisely* the ones observed in the corresponding **emission spectrum**. In absorption, certain photons of light are absorbed, giving up energy to the atom and enabling the electron to move from a lower (ℓ) to a higher (u) stationary state. Equations (4.27) and (4.30) describe the frequencies and wavelengths of the absorbed photons. The atom will remain in the excited state for only a very short time (on the order of 10^{-10} s) before emitting a photon and returning to a lower stationary state. Thus, at ordinary temperatures practically all hydrogen atoms exist in the lowest possible energy state, $n = 1$, and only the absorption spectral lines of the Lyman series are

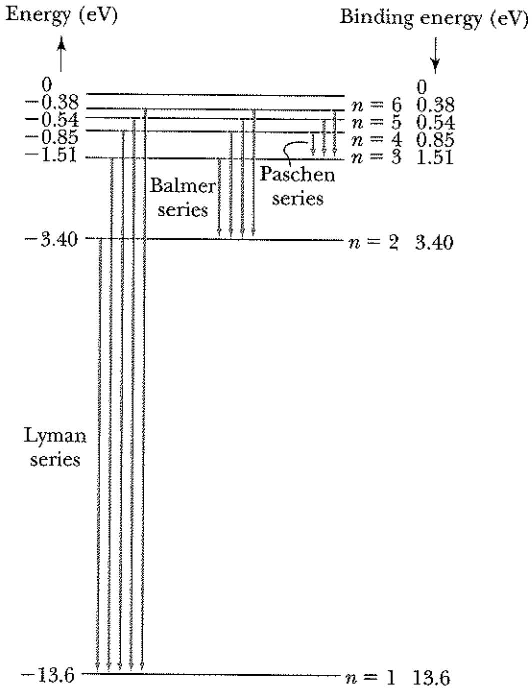


Figure 4.16 Transitions between many of the stationary states in the hydrogen atom are indicated. Transitions (ultraviolet) to the $n = 1$ state from the higher-lying states are called the Lyman series. The transitions shown to the $n = 2$ state (Balmer series) were discovered first because they are in the visible wavelength range. The Paschen series (transitions to $n = 3$) are in the infrared. The energies of each state as well as the binding energies are noted.

normally observed. However, these lines are not in the visible region. The sun produces electromagnetic radiation over a wide range of wavelengths, including the visible region. When sunlight passes through our atmosphere, hydrogen atoms in water vapor absorb the wavelengths of the Balmer series (visible region), and the absorption spectrum has dark lines at the known wavelengths of the Balmer series.

We can determine the electron's velocity in the Bohr model from Equations (4.22b) and (4.23).

$$v_n = \frac{n\hbar}{mr_n} = \frac{n\hbar}{mn^2a_0} = \frac{1}{n}\frac{\hbar}{ma_0} \tag{4.31}$$

or

$$v_n = \frac{1}{n}\frac{e^2}{4\pi\epsilon_0\hbar}$$

The value of v_1 is $\hbar/ma_0 = 2.2 \times 10^6$ m/s, which is less than 1% of the speed of light. We define the dimensionless quantity ratio of v_1 to c as

$$\alpha \equiv \frac{v_1}{c} = \frac{\hbar}{ma_0c} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \tag{4.32}$$

Fine structure constant

This ratio is called the **fine structure constant.** It appears often in atomic structure calculations.

We insert a word of caution at this point. Bohr's atomic model of quantized energy levels represented a significant step forward in understanding the structure of the atom. Although it had many successes, we know now that, in principle, it is wrong. We will discuss some of its successes and failures in the next section and will discuss the correct quantum theory in Chapter 6. Nevertheless, Bohr's atomic model is useful in our first attempt in understanding the structure of the atom.