4. Similarly, consider a vector that points exclusively along the x' axis, with components $(\Delta x', 0)$. Its components along the y and x axes stand to each other in the ratio

$$\Delta y/\Delta x = -S_r$$

This information plus the invariance of the distance

$$(\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + 0$$

lead by the same type of reasoning to the result

$$\Delta x = (1 + S_r^2)^{-1/2}\Delta x'$$
$$\Delta y = -S_r(1 + S_r^2)^{-1/2}\Delta x'$$

These expressions confirm the remaining two coefficients in Eqs. 19 for a Euclidean transformation.

Relative slope S_r (Euclid) compared with relative velocity β_r (Lorentz)

In summary, the covariant transformation in Euclidean geometry from $(\Delta x', \Delta y')$ to $(\Delta x, \Delta y)$ is clearly analogous to the transformation from $(\Delta x', \Delta t')$ to $(\Delta x, \Delta t)$ in the Lorentz geometry of the real physical world. The *slope* S_r of the axis of one coordinate system relative to the corresponding axis of the other system is analogous to the velocity β_r of one inertial reference frame relative to the other. The ratios between the two sides of a right triangle and its hypotenuse in Euclidean geometry

$$\frac{1}{(1 + S_r^2)^{1/2}} \quad \text{and} \quad \frac{S_r}{(1 + S_r^2)^{1/2}}$$

are replaced in Lorentz geometry by the expressions

$$\frac{1}{(1 - \beta_r^2)^{1/2}} \quad \text{and} \quad \frac{\beta_r}{(1 - \beta_r^2)^{1/2}}$$

The minus sign in the expression $(1 - \beta_r^2)^{1/2}$ contrasts with the plus sign in $(1 + S_r^2)^{1/2}$. The negative sign originates from the minus sign in the expression for the interval in Lorentz geometry.

9. The Velocity Parameter

Additivity of angles suggests looking for additive velocity parameter

Have we finished? We have determined how to go from a knowledge of the components of a separation in one reference frame to a calculation of the components of the separation in another reference frame. In brief, we have written down the covariant law of connection of components both for a Lorentz transformation ("transformation in x, t plane") and for a rotation ("transformation in x, y plane"). In one, the formulas contain the parameter β_r (the relative velocity); in the other, the parameter S_r (the relative slope). However, neither of these parameters provides the simplest way to describe the relation between two coordinate systems. It is desirable to replace both β_r and S_r by more natural parameters. We can find better means to describe a velocity and a rotation! *Angle* is the best measure of rotation. Similarly, a certain *velocity parameter* θ, yet to be defined, is the most convenient measure of velocity. The usefulness and meaning of this velocity parameter in describing

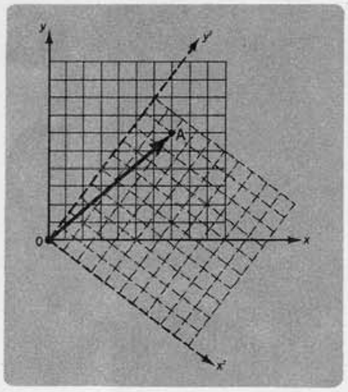


FIG. 26, PAGE 45

velocities will best be appreciated by asking here: Why is an angle a more convenient parameter than a slope for measuring a rotation?

And the answer is: Because *angles are additive and slopes are not*. What does this statement mean? Refer to Fig. 26. The vector OA is inclined to the y' axis. This inclination may be described by the slope S' (the number of units of distance in the x' direction per unit of distance in the y' direction). In the example this slope is

$$S' = 2/9$$

In contrast, the vector OA is inclined to the y axis by a slope

$$S = 7/6$$

Further, the y' axis is inclined to the y axis by a slope

$$S_r = 3/4$$

Question: Is the following law of addition of slope correct?

$$\begin{pmatrix}\text{slope of OA}\\ \text{relative to}\\ y \text{ axis}\end{pmatrix} \overset{?}{=} \begin{pmatrix}\text{slope of OA}\\ \text{relative to}\\ y' \text{ axis}\end{pmatrix} + \begin{pmatrix}\text{slope of } y' \text{ axis}\\ \text{relative to}\\ y \text{ axis}\end{pmatrix}$$

Slopes in Euclidean geometry are not additive

Test ("experimental mathematics"):

$$(7/6) \overset{?}{=} (2/9) + (3/4)$$
$$(42/36) \overset{?}{=} (8/36) + (27/36)$$
$$42 \overset{?}{=} 8 + 27 = 35 \qquad \text{No!}$$

Conclusion: Slopes are not additive! Question: If slopes are not additive and S is not equal to the sum of S' and S_r, what then is the correct way to deduce the slope S from S' and S_r? Answer:

$$\begin{pmatrix}\text{slope of OA}\\ \text{relative to}\\ y \text{ axis}\end{pmatrix} = S = \Delta x/\Delta y \qquad \text{(by definition of slope)}$$

$$= \frac{(1 + S_r^2)^{-1/2}\Delta x' + S_r(1 + S_r^2)^{-1/2}\Delta y'}{-S_r(1 + S_r^2)^{-1/2}\Delta x' + (1 + S_r^2)^{-1/2}\Delta y'} \qquad \text{(from Eqs. 19)}$$

$$= \frac{\Delta x' + S_r\Delta y'}{-S_r\Delta x' + \Delta y'} \qquad \text{(by eliminating } (1 + S_r^2)^{-1/2} \text{ from numerator and denominator)}$$

$$= \frac{(\Delta x'/\Delta y') + S_r}{-S_r(\Delta x'/\Delta y') + 1} \qquad \text{(by dividing numerator and denominator by } \Delta y'\text{)}$$

Thus finally,

$$S = \frac{S' + S_r}{1 - S'S_r} \tag{20}$$

In other words, two slopes S' and S_r can be treated as additive only when the product $S'\,S_r$ in the denominator can be neglected in comparison with unity.

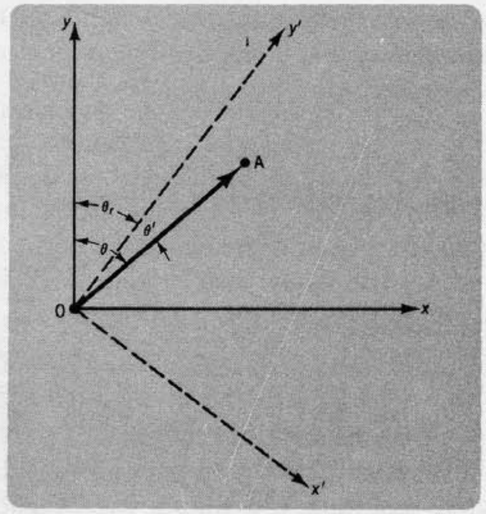


Fig. 28. The *angle* is a convenient way to measure inclination between y axis and y' axis—convenient because angles satisfy a simple law of addition: $\theta = \theta' + \theta_r$.

Angles ARE additive

Since slopes are not additive and are thus not the convenient way to measure the inclination of two coordinate systems, what then is a more suitable way to measure this inclination? Answer: The *angle* between the y and y' axes. Why? Because angle does satisfy a simple law of addition (Fig. 28).

$$\begin{pmatrix}\text{angle of OA}\\ \text{relative to}\\ y\text{ axis}\end{pmatrix} = \begin{pmatrix}\text{angle of OA}\\ \text{relative to}\\ y'\text{ axis}\end{pmatrix} + \begin{pmatrix}\text{angle of } y' \text{ axis}\\ \text{relative to}\\ y\text{ axis}\end{pmatrix}$$

or

(21) $$\theta = \theta' + \theta_r$$

The existence of this relation makes the angle the simple measure of inclination.

What is the relation between this new measure of inclination and the old measure, the slope S_r of the y' axis relative to the y axis? Answer:

(22) $S_r = \tan \theta_r$ (from the definition of tangent in trigonometry; see Fig. 29)

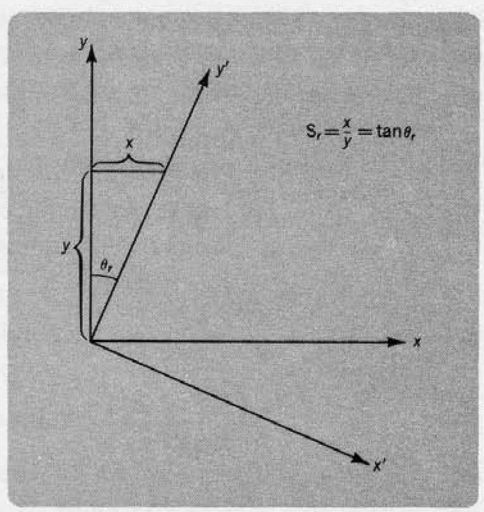


Fig. 29. Relation between relative slope S_r of corresponding axes of two Euclidean coordinate systems and the angle θ_r between these axes.

Euclidean law of addition of slopes

Question: How does one understand the law of addition of slopes when one recognizes that a slope is the tangent of an angle? Answer:

$$\tan \theta = \tan (\theta' + \theta_r) \qquad \text{(additivity of angles)}$$

$$(23) \qquad = \frac{\tan \theta' + \tan \theta_r}{1 - \tan \theta' \tan \theta_r} \qquad \text{(trigonometry)}$$

or

$$S = \frac{S' + S_r}{1 - S'S_r} \qquad \text{(tangents measure slopes)}$$

Comparison of the complicated law of addition of tangents, or slopes, and the simple law of addition of angles, $\theta = \theta' + \theta_r$, confirms that angles provide the simplest measure of rotations.

Law of addition of velocities

What is the simplest measure of velocity? Not velocity itself. Velocity itself does not satisfy a simple law of addition. What is the law of addition of velocities? Let a bullet be fired forward at a velocity β' in the rocket frame of reference (Fig. 30).

$$\beta' = \frac{\begin{pmatrix}\text{number of meters of advance}\\ \text{in the } x' \text{ direction for each}\end{pmatrix}}{\begin{pmatrix}\text{meter of advance in the read-}\\ \text{ings } t' \text{ of the rocket clocks}\end{pmatrix}} = (\Delta x'/\Delta t')$$

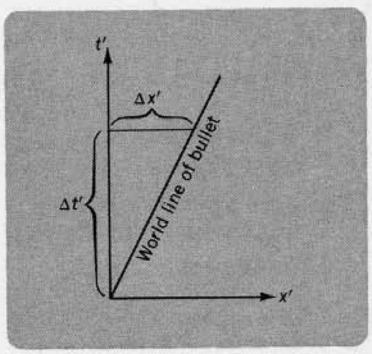


Fig. 30. World line of a bullet plotted in the rocket spacetime diagram. The bullet is fired forward with a velocity $\beta' = \Delta x'/\Delta t'$ in the rocket frame.

The rocket is moving at the velocity β_r relative to the laboratory. What is the velocity β of the *bullet* relative to the laboratory—as measured with the laboratory latticework of clocks? Answer: The velocity is

$$\beta = \frac{\begin{pmatrix}\text{number of meters of advance}\\ \text{in the } x \text{ direction for each}\end{pmatrix}}{\begin{pmatrix}\text{meter of advance in the read-}\\ \text{ings } t \text{ of the laboratory clocks}\end{pmatrix}} = (\Delta x/\Delta t)$$

$$= \frac{(1-\beta_r^2)^{-1/2}\Delta x' + \beta_r(1-\beta_r^2)^{-1/2}\Delta t'}{\beta_r(1-\beta_r^2)^{-1/2}\Delta x' + (1-\beta_r^2)^{-1/2}\Delta t'} \qquad \text{(Lorentz transformation, Eqs. 16)}$$

$$= \frac{\Delta x' + \beta_r \Delta t'}{\beta_r \Delta x' + \Delta t'} \qquad \text{(by eliminating } (1-\beta_r^2)^{-1/2} \text{ from numerator and denominator)}$$

$$= \frac{(\Delta x'/\Delta t') + \beta_r}{\beta_r(\Delta x'/\Delta t') + 1} \qquad \text{(by dividing numerator and denominator by } \Delta t'\text{)}$$

Thus finally,

(24) $$\beta = \frac{\beta' + \beta_r}{1 + \beta'\beta_r} \quad \text{(law of addition of velocities)}$$

In other words, velocities are not additive. *Limiting case, for low velocities only:* The two velocities β' and β_r can be treated as additive (to a certain level of accuracy) when the product $\beta'\beta_r$ in the denominator is negligibly small compared to unity (to that same level of accuracy, whether this level of accuracy is 1 part in 10 or 1 part in 10^6). *Example of lack of additivity of velocities:* The rocket is already going at 3/4 of the speed of light when it fires a bullet. The bullet itself moves at 3/4 of the speed of light relative to the rocket. What is the speed of the bullet relative to the laboratory? Answer: Not $(3/4) + (3/4) = 1.5$ times the speed of light, but instead

$$\beta = \frac{(3/4) + (3/4)}{1 + (3/4)(3/4)} = \frac{(3/2)}{(25/16)} = \frac{24}{25} = 0.96$$

(meters of laboratory distance per meter of travel time of light in the laboratory). Thus the relativistic law of addition of velocities (24) ensures that no object can ever be propelled at a speed as great as the speed of light.

Considering that velocities themselves are not additive, we propose to find a new measure of velocity, a "velocity parameter" θ, which *is* additive; thus,

$$\begin{pmatrix}\text{velocity parameter}\\ \text{of bullet relative}\\ \text{to laboratory}\end{pmatrix} = \begin{pmatrix}\text{velocity parameter}\\ \text{of bullet relative}\\ \text{to rocket}\end{pmatrix} + \begin{pmatrix}\text{velocity parameter}\\ \text{of rocket relative}\\ \text{to laboratory}\end{pmatrix}$$

Velocity parameter: defined to be additive!

or

(25) $$\theta = \theta' + \theta_r$$

This parameter θ will be quite different in meaning from the angle that describes rotations. The velocity parameter cannot be represented as a simple angle in any diagram, and for a very good reason. Distances between points on a piece of paper are governed by the laws of Euclidean geometry. In contrast, the intervals between the events of the physical world are controlled by the Lorentz geometry of spacetime. But the impossibility of freezing moving bullets and ticking clocks onto a piece of paper does not deprive these lively objects of one iota of their reality. And the further impossibility of depicting the additivity of the velocity parameter θ on a page does not discourage us, but merely invites us to look at the real world of fast particles and high-energy physics to see the law of addition of velocity parameters in action. This law of addition of velocity parameters, $\theta = \theta' + \theta_r$, is every bit as real as the law of addition of angles of rotation.

What is the connection between velocity β and velocity parameter θ? The appropriate formula is analogous to the formula for slope in terms of angle (slope = tangent of angle). It has the form

Velocity is hyperbolic tangent of velocity parameter

(26) $$\beta = \tanh\theta$$

Here "tanh" is read "hyperbolic tangent." The hyperbolic tangent function, as well as the hyperbolic sine and cosine functions, sinh θ and cosh θ (with

$\tanh \theta = \sinh \theta/\cosh \theta$), forms a standard part of mathematics. Tables of all three functions are given in every comprehensive compilation of tables. Formal definitions of these functions are presented in Table 8. Nevertheless, we need no knowledge of these tables and this mathematical literature. All that we want to know about the function $\tanh \theta$ can—naturally enough—be found from its very definition. Two properties define it: (*a*) It must correctly describe the law of addition of velocities. Out of the relation

$$\beta = \frac{\beta' + \beta_r}{1 + \beta'\beta_r}$$

and the demand $\theta = \theta' + \theta_r$ we read the law of addition

$$\tanh(\theta' + \theta_r) = \tanh \theta = \frac{\tanh \theta' + \tanh \theta_r}{1 + \tanh \theta' \tanh \theta_r} \tag{27}$$

(from equation of definition 26)

(*b*) For low velocities the velocity parameter θ must reduce to the usual measure of velocity, β. This requirement means that $\tanh \theta$ must become arbitrarily close to θ itself for small θ. We recall that the ordinary tangent of an ordinary angle reduces to the angle itself for small angles, provided that the angle is measured in radians. When the angle is measured in degrees, there is a correction factor, $\pi/180°$. Similarly, the velocity parameter can here be measured in a variety of units, analogous to degrees and minutes, but the simplest unit is that in which $\tanh \theta \xrightarrow[\text{small } \theta]{} \theta$. We can call this unit the hyperbolic radian (dimensionless).

How can the connection between velocity parameter and velocity be found from the principles of (*a*) additivity and (*b*) $\tanh \theta = \theta$ for small velocity parameters?

Answer: (1) Start with a velocity parameter θ small enough so that $\tanh \theta$ can be identified with θ to some appropriate level of accuracy. Thus, write

Constructing table of hyperbolic tangents

$$\tanh 0.01 = 0.01$$

as the first entry in the desired table of hyperbolic tangents.

(2) Get the next entry by using the law of addition (27); thus,

$$\tanh 0.02 = \tanh(0.01 + 0.01) = \frac{\tanh 0.01 + \tanh 0.01}{1 + (\tanh 0.01)(\tanh 0.01)} = \frac{0.01 + 0.01}{1 + 0.0001} \tag{28}$$

(3) At this point a decision has to be made about the accuracy of the number work. Why not take tanh 0.02 to have the value 0.02 just as we took tanh 0.01 to have the value 0.01? Because there is a correction term of 0.0001 in the denominator of (28). Its presence implies that 0.02 will depart from the correct value of tanh 0.02 by roughly 1 part of 10^4. We here and now decide that we will calculate all tanh values correct to one part in 10^4. We will therefore want to include the 0.0001 correction in the denominator. But if we have to make such a correction in evaluating tanh 0.02, why did we not make such a correction in evaluating tanh 0.01? Because that correction would have been still smaller. In other words, the difference between tanh 0.01 and 0.01 can be

neglected when one is concerned to have his results correct to "only" one part in 10^4. To this accuracy we thus finally have

$$\tanh 0.02 = \frac{0.020000}{1.0001} = 0.019998$$

(4) Now ask for the value of tanh 0.04

$$\tanh 0.04 = \tanh(0.02 + 0.02) = \frac{\tanh 0.02 + \tanh 0.02}{1 + (\tanh 0.02)(\tanh 0.02)}$$

$$= \frac{2 \times 0.019998}{1 + (0.019998)^2} = 0.039980$$

The correction term in the denominator now affects the numerical value of the result by about 4 parts in 10^4. Nevertheless the result is good to about 1 part in 10^4. The result has been obtained by using a correct formula (Eq. 27) to combine hyperbolic tangent values, which were themselves correct to 1 part in 10^4.

(5) We construct further entries in the hyperbolic tangent table by the same type of combinatorial procedure. Thus, from a knowledge of tanh 0.04 and tanh 0.01 we can calculate tanh 0.05 = tanh (0.04 + 0.01). We go on to get tanh 0.1, tanh 0.2, and tanh 0.4; then tanh 0.5 = tanh (0.4 + 0.1). Similarly we calculate tanh 1, tanh 2, and any other values we want. In this way we find the results summarized in Fig. 31.

Contrast between velocity parameter and ordinary angle

Two features of the velocity parameter stand out at once from Fig. 31, quite apart from any details of the numbers. First, the slope of the curve of $\tanh\theta$ versus θ goes to unity at small θ—another way of saying that the velocity, $\beta = \tanh\theta$, and the velocity parameter θ approach equality at small θ. Second, the velocity parameter θ goes to indefinitely large positive (or negative) values as the velocity, $\beta = \tanh\theta$, itself approaches plus (or minus) unity. In other

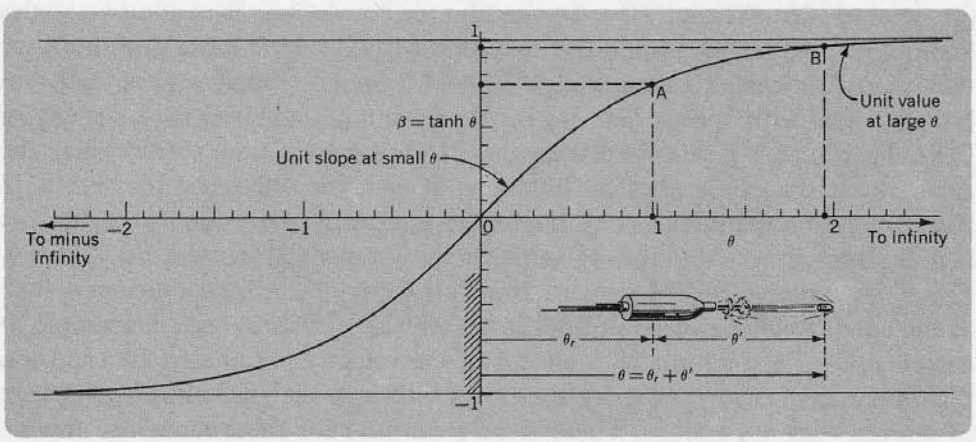


Fig. 31. Relation between velocity parameter θ and the velocity $\beta = \tanh\theta$ as determined directly from the law of addition

$$\tanh(\theta_1 + \theta_2) = \frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1 \tanh\theta_2}$$

as described in the text. Example: A bullet is fired at a speed $\beta' = 0.75$ from a rocket traveling at a speed $\beta_r = 0.75$. Find the speed β of the bullet relative to the laboratory. The velocity is not additive but the velocity parameter is. From the graph (point A) read off $\theta' = \theta_r = 0.973$. Add: $\theta = \theta' + \theta_r = 1.946$. For this value of the velocity parameter read off from the graph (point B) the result $\beta = 0.96$. The same result is obtained in the text in another way (p. 51).

words any values for the velocity parameter are conceivable, extending over the entire range from $\theta = -\infty$ to $\theta = +\infty$. The contrast between "hyperbolic angles" or velocity parameters, with this infinite extent of variation, and ordinary angles is evident. An ordinary angle leads to nothing new after it has increased through the finite range from 0 to 2π radians.

Velocity parameter and invariant speed of light

Velocity parameters and the law of addition of velocities—what connection have these ideas with the elementary physical observations that forced on physics the spacetime point of view? The most direct connection possible: From the observations—and from what was known even in 1905 about electromagnetic waves—Einstein was led to conclude that the speed of light is the same in all inertial reference frames. In other words—to translate into the language of idealized experiments—a photon shot with the speed of light from a fast rocket travels relative to the laboratory with a speed that is *also* equal to that of light. In the language of velocity parameters, the rocket has a finite parameter θ_r; but the photon ($\beta' = 1$) has an infinite velocity parameter ($\theta' = \infty$; Fig. 31, upper right, asymptotic limit). Add a finite number to infinity and end up with infinity for the sum $\theta = \theta' + \theta_r$. Thus the speed of the photon in the laboratory frame, $\beta = \tanh\theta = \tanh\infty = 1$, again agrees with the speed of light. We have come full circle, back to the starting idea of relativity: that the speed of light has the same value in all frames of reference.

Simplicity of velocity parameter

We conclude that the velocity parameter with its simple law of addition, $\theta = \theta' + \theta_r$, is the natural way to measure velocities. Then why does one not have a direct intuitive grasp of this measure of velocity? Why is not the hyperbolic angle as familiar to every school child as the ordinary angle? The answer is simple. Everyday experience deals with angles of all sizes, large and small. Therefore no one would be so naive as to add slope $S' = 1$ (angle of 45°) to slope $S_r = 1$ (another angle of 45°) and expect to get slope $S = S' + S_r = 2$ (angle of 63° 26′. Wrong!). One knows that the correct way is to add two angles (sum: 45° + 45° = 90°; slope $S = \infty$). But everyday experience does not deal with velocities close to the speed of light. Motor cars, real rockets, and real bullets travel with speeds that are extremely small compared to the speed of light. Therefore it is not surprising that it took a long time to recognize the truth about spacetime physics. But now, at last, the difference that exists in nature between the law of combination of velocities (the complicated Eq. 24) and the law of combination of velocity parameters (the simple Eq. 21: $\theta = \theta' + \theta_r$) is understood. Moreover, previously perplexing observations—such as the equality of the speed of light in all reference frames—become simple to describe when one adopts the concept of the velocity parameter. In addition, this parameter—and everything that goes with it in the spacetime description of physics—are necessities. There is no substitute for these ideas for anyone who wants to look upon the structure of the physical world as that four-dimensional world really is. More and more this necessity becomes clear as electronuclear machines and high velocity particles become part of the fabric of modern civilization.

There is no way around it! The velocity parameter provides the simple way to measure speed, as the ordinary angle provides the simple way to measure inclination. Having accepted this conclusion, what profit can we draw from it in the form of a simpler way to describe a Lorentz transformation?

Ask first, by way of orientation, the analogous question about the Euclidean geometry of the xy plane. Does the formula (Eqs. 19) for calculating one set of coordinates in terms of the other

Simplify Euclidean transformation using angle

$$\begin{aligned}\Delta x &= (1 + S_r^2)^{-1/2}\,\Delta x' + S_r(1 + S_r^2)^{-1/2}\,\Delta y' \\ \Delta y &= -S_r(1 + S_r^2)^{-1/2}\,\Delta x' + (1 + S_r^2)^{-1/2}\,\Delta y'\end{aligned}$$

reduce in complexity when one expresses the relative slope S_r of the y and y' axes in terms of the ordinary angle θ_r? Answer: The coefficients in the rotational transformation become

$$(1 + S_r^2)^{-1/2} = (1 + \tan^2\theta_r)^{-1/2} = \left(\frac{\cos^2\theta_r + \sin^2\theta_r}{\cos^2\theta_r}\right)^{-1/2} = \left(\frac{1}{\cos^2\theta_r}\right)^{-1/2} = \cos\theta_r$$

and

$$S_r(1 + S_r^2)^{-1/2} = \tan\theta_r\cos\theta_r = \frac{\sin\theta_r}{\cos\theta_r}\cos\theta_r = \sin\theta_r$$

Therefore the transformation equation itself takes the form

$$\begin{aligned}\Delta x &= \Delta x'\cos\theta_r + \Delta y'\sin\theta_r \\ \Delta y &= -\Delta x'\sin\theta_r + \Delta y'\cos\theta_r\end{aligned} \tag{29}$$

and we conclude: The relation between old and new coordinates takes its simplest form when the coefficients in the covariant transformation are expressed as "trigonometric," or "circular," functions of the angle of rotation.

Now turn to the Lorentz transformation written in terms of the relative velocity

Simplify Lorentz transformation using velocity parameter

$$\begin{aligned}\Delta x &= (1 - \beta_r^2)^{-1/2}\,\Delta x' + \beta_r(1 - \beta_r^2)^{-1/2}\,\Delta t' \\ \Delta t &= \beta_r(1 - \beta_r^2)^{-1/2}\,\Delta x' + (1 - \beta_r^2)^{-1/2}\,\Delta t'\end{aligned}$$

How does this pair of equations look when expressed in terms of the improved measure of velocity, θ_r? Answer: Recall the connection between the velocity β_r and the velocity parameter

$$\beta_r = \tanh\theta_r$$

Note that the coefficients in the Lorentz transformation depend upon β_r, and by that very token are fixed by our choice of θ_r. These coefficients have the form

$$(1 - \beta_r^2)^{-1/2} = (1 - \tanh^2\theta_r)^{-1/2} \tag{30}$$

and

$$\beta_r(1 - \beta_r^2)^{-1/2} = \tanh\theta_r\,(1 - \tanh^2\theta_r)^{-1/2} \tag{31}$$

These expressions have a rather complicated appearance. Nevertheless, they are well defined. For any given value of θ_r we know how to find the value of $\tanh\theta_r$ (Fig. 31 and corresponding text). From this value of $\tanh\theta_r$ we can evaluate (30) and (31) with any desired accuracy for any given value of the velocity parameter. These two functions of θ_r have such importance that they have received names of their own in the literature on hyperbolic functions. To give the functions in question their standard names in no way decreases our ability to find the values of these functions at any time we please through our own efforts and without reference to any treatises or tables. Therefore we accept and use the standard names hereafter:

$$(1 - \tanh^2\theta_r)^{-1/2} = \cosh\theta_r = \begin{pmatrix}\text{hyperbolic}\\ \text{cosine of } \theta_r\end{pmatrix}\ \text{names; nothing}$$
$$\tanh\theta_r\,(1 - \tanh^2\theta_r)^{-1/2} = \sinh\theta_r = \begin{pmatrix}\text{hyperbolic}\\ \text{sine of } \theta_r\end{pmatrix}\ \text{but names!}$$

Using this nomenclature, we find that the equations of the Lorentz transformation take the following form

Lorentz transformation using velocity parameter

$$\begin{aligned}\Delta x &= \Delta x' \cosh\theta_r + \Delta t' \sinh\theta_r\\ \Delta t &= \Delta x' \sinh\theta_r + \Delta t' \cosh\theta_r\end{aligned} \tag{32}$$

and we conclude: The relation between old and new coordinates takes its simplest form when the coefficients in the transformation are expressed as hyperbolic functions of the velocity parameter θ_r of the relative motion. Moreover, expressed in terms of hyperbolic sines and cosines, the Lorentz transformation takes a form that corresponds even more closely than before to the standard trigonometric form (29) for a rotational transformation.

What can one do to grasp and feel the properties of the hyperbolic functions that appear in the Lorentz transformation? The two most interesting and important properties of these functions follow immediately from the definitions (Eqs. 30 and 31). First, the ratio of the two hyperbolic functions has the value

$$\sinh\theta_r/\cosh\theta_r = \tanh\theta_r \tag{33}$$

in complete analogy to the corresponding relation for circular functions. Second, the difference between the squares of the two hyperbolic functions is

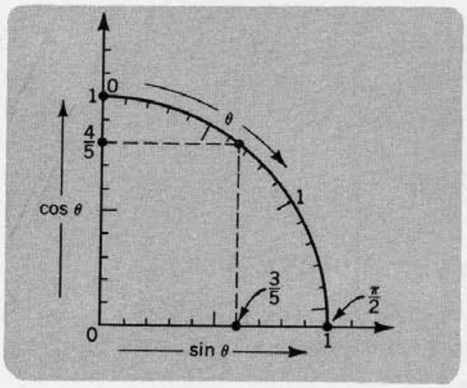
$$\cosh^2\theta_r - \sinh^2\theta_r = \frac{1}{(1-\tanh^2\theta_r)} - \frac{\tanh^2\theta_r}{(1-\tanh^2\theta_r)} = \frac{1-\tanh^2\theta_r}{1-\tanh^2\theta_r} = 1 \tag{34}$$

Contrast this formula with the analogous relation for trigonometric functions

$$\cos^2(\text{angle}) + \sin^2(\text{angle}) = 1 \tag{35}$$

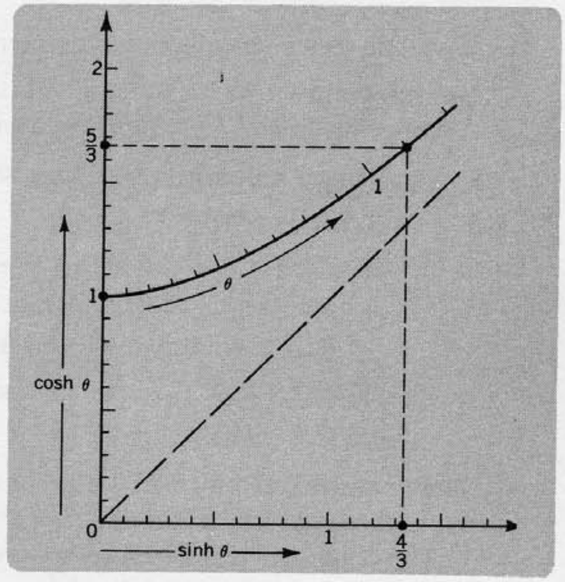
Circular functions compared with hyperbolic functions

Equations 34 and 35 admit a simple geometrical interpretation. Plot sin (angle) as the horizontal coordinate and cos (angle) as the vertical coordinate in Fig. 32. Then Eq. 35 is the equation of a circle of unit radius—whence the often used term "circular functions" for the sine and the cosine. In contrast, (34) is the equation of a hyperbola (Fig. 33)—hence the word "hyperbolic functions." The positive sign in the expression $\cos^2 + \sin^2 = 1$ has its origin in the way x components and y components of a vector are combined to obtain the square of the length of that vector. And why the minus sign in $\cosh^2\theta - \sinh^2\theta = 1$? Because the square of a spacetime interval is given by the square of the separation in time *diminished* by the square of the separation in space.



$$\boxed{\cos^2 \theta + \sin^2 \theta = 1.}$$

Fig. 32. Circle representing cos θ versus sin θ for circular functions. Example: $(3/5)^2 + (4/5)^2 = 1$.



$$\boxed{\cosh^2 \theta - \sinh^2 \theta = 1.}$$

Fig. 33. Hyperbola representing cosh θ versus sinh θ for hyperbolic functions. Example: $(5/3)^2 - (4/3)^2 = 1$.

Confirmation: Euclidean transformation preserves distance invariant

The distinction between the plus sign in $\cos^2 + \sin^2 = 1$ and the minus sign in $\cosh^2 \theta - \sinh^2 \theta = 1$ has to do with the contrast between the length in Euclidean geometry and the interval in Lorentz geometry. Look at this point more closely for the two kinds of geometry in turn. In Euclidean geometry reconfirm that the *covariant* transformation of *coordinates* (Eqs. 29)—now expressed in terms of circular functions rather than slope—guarantees the principle of *invariance of length.* For this purpose calculate $(\text{length})^2 = (\Delta x)^2 + (\Delta y)^2$ from (Eqs. 29) and find

$$\begin{aligned}
(\text{length})^2 &= (\Delta x)^2 + (\Delta y)^2 \\
&= (\Delta x' \cos \theta_r + \Delta y' \sin \theta_r)^2 + (-\Delta x' \sin \theta_r + \Delta y' \cos \theta_r)^2 \\
&= (\Delta x')^2 \cos^2 \theta_r \cancel{+ 2(\Delta x')(\Delta y') \cos \theta_r \sin \theta_r} + (\Delta y')^2 \sin^2 \theta_r \\
&\quad + (\Delta x')^2 \sin^2 \theta_r \cancel{- 2(\Delta x')(\Delta y') \sin \theta_r \cos \theta_r} + (\Delta y')^2 \cos^2 \theta_r \\
&= [(\Delta x')^2 + (\Delta y')^2] (\cos^2 \theta_r + \sin^2 \theta_r) \\
&= (\Delta x')^2 + (\Delta y')^2
\end{aligned}$$

confirming the invariance of the expression for length. Note the importance of the relation

$$\cos^2 \theta_r + \sin^2 \theta_r = 1$$

in connecting the ideas of covariance (transformation of coordinates associated with different orientation of two coordinate systems) and invariance (length the same in both systems).

Confirmation: Lorentz transformation preserves interval invariant

The connection between covariance and invariance in Lorentz geometry rests equally clearly on the relation

$$\cosh^2 \theta_r - \sinh^2 \theta_r = 1$$

This one sees on calculating any interval whether spacelike or timelike

$$\begin{aligned}\begin{pmatrix}\text{interval of}\\ \text{proper distance}\end{pmatrix}^2 &= -\begin{pmatrix}\text{interval of}\\ \text{proper time}\end{pmatrix}^2\\
&= (\text{space separation})^2 - (\text{time separation})^2\\
&= (\Delta x)^2 - (\Delta t)^2\\
&= (\Delta x' \cosh \theta_r + \Delta t' \sinh \theta_r)^2 - (\Delta x' \sinh \theta_r + \Delta t' \cosh \theta_r)^2\\
&= (\Delta x')^2 \cosh^2 \theta_r + \cancel{2(\Delta x')(\Delta t') \cosh \theta_r \sinh \theta_r} + (\Delta t')^2 \sinh^2 \theta_r\\
&\quad - [(\Delta x')^2 \sinh^2 \theta_r + \cancel{2(\Delta x')(\Delta t') \sinh \theta_r \cosh \theta_r} + (\Delta t')^2 \cosh^2 \theta_r]\\
&= [(\Delta x')^2 - (\Delta t')^2]\,(\cosh^2 \theta_r - \sinh^2 \theta_r)\\
&= (\Delta x')^2 - (\Delta t')^2\end{aligned}$$

Here one sees reconfirmed in the simplest way possible that a Lorentz transformation preserves the invariance of the expression for the interval.

Inverse Lorentz transformation

The Lorentz transformation—we have now confirmed in all detail—translates from the specialized language of rocket coordinates (x', t') to the specialized language of laboratory coordinates (x, t). Moreover, the scheme of translation is consistent at every point with the universal language of intervals (consistency of covariant description of spacetime physics with invariant description of spacetime physics). However, we need still more: The typical Turkish-English dictionary is bound together with an English-Turkish dictionary—where is the second "relativity dictionary?" How can we go backwards from a knowledge of x and t to a knowledge of x' and t'? If one dictionary is provided by the formulas

$$\begin{aligned} x &= x' \cosh \theta_r + t' \sinh \theta_r \\ t &= x' \sinh \theta_r + t' \cosh \theta_r \end{aligned} \tag{36}$$

what are the formulas for translation backwards from laboratory records to rocket records? *Answer:* The Lorentz transformation "inverse" to Eqs. 36 is

$$\begin{aligned} x' &= x \cosh \theta_r - t \sinh \theta_r \\ t' &= -x \sinh \theta_r + t \cosh \theta_r \end{aligned} \tag{37}$$

Proof: Substitute these expressions for x' and t' into Eqs. 36 and verify that identities result (an English word translated into Turkish and then back into English comes out as the original word provided that the one dictionary is the true inverse of the other!).

In the following table, formal definitions of the hyperbolic functions and some of the relations that they satisfy are presented in parallel with similar definitions and relations for circular functions. In this table e is the base of the natural logarithms and has the numerical value 2.718281 . . . . The symbol i stands for the square root of minus one, so that $i^2 = -1$. The usual rules for addition and multiplication of exponents apply to exponents containing i. The angle θ is expressed in circular or hyperbolic radians (*not* degrees). The ex- expression 4!, for instance, means *four factorial*: $4 \times 3 \times 2 \times 1$. To understand these relations derive lines 7 to 13 from the definitions in lines 1 to 6 on each side of the table and show qualitatively how the graphs of Figs. 32 and 33 follow from these relations. Note especially the differences in *sign* between the two sides of the table.

Table 8. Circular and hyperbolic functions.

Circular functions	*Hyperbolic functions*
DEFINITIONS	
1. $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$	1. $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$
2. $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	2. $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$
3. $\tan\theta = \frac{\sin\theta}{\cos\theta}$	3. $\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$
4. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$	4. $\sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots$
5. $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$	5. $\cosh\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots$
6. $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$	6. $\tanh\theta = \theta - \frac{\theta^3}{3} + \frac{2}{15}\theta^5 - \cdots$
RELATIONS	
7. $\sin(-\theta) = -\sin\theta$	7. $\sinh(-\theta) = -\sinh\theta$
8. $\cos(-\theta) = \cos\theta$	8. $\cosh(-\theta) = \cosh\theta$
9. $\tan(-\theta) = -\tan\theta$	9. $\tanh(-\theta) = -\tanh\theta$
10. $\boxed{\cos^2\theta + \sin^2\theta = 1}$	10. $\boxed{\cosh^2\theta - \sinh^2\theta = 1}$
11. $\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$	11. $\sinh(\theta_1 + \theta_2) = \sinh\theta_1\cosh\theta_2 + \cosh\theta_1\sinh\theta_2$
12. $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$	12. $\cosh(\theta_1 + \theta_2) = \cosh\theta_1\cosh\theta_2 + \sinh\theta_1\sinh\theta_2$
13. $\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2}$	13. $\tanh(\theta_1 + \theta_2) = \frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1\tanh\theta_2}$
POOR MAN'S QUICK RECIPES	
For small θ: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$	For small θ: $\sinh\theta \approx \theta$, $\tanh\theta \approx \theta$
Example: $\theta = 0.1$	*Example:* $\theta = 0.1$
Poor man's recipe: $\sin\theta \approx 0.1$, $\tan\theta \approx 0.1$	Poor man's recipe: $\sinh\theta \approx 0.1$, $\tanh\theta \approx 0.1$
Accurate values: $\sin\theta = 0.0998$, $\tan\theta = 0.1003$	Accurate values: $\sinh\theta = 0.1002$, $\tanh\theta = 0.0997$
	For large θ: $\sinh\theta \approx e^\theta/2$, $\cosh\theta \approx e^\theta/2$
	Example: $\theta = 3$ $e^\theta \approx 20$
	Poor man's recipe: $\sinh\theta \approx 10$, $\cosh\theta \approx 10$
	Accurate values: $\sinh\theta = 10.018$, $\cosh\theta = 10.068$