A Non-Residually Solvable Hyperlinear One-Relator Group

Jon P. Bannon

ABSTRACT. In this short note, we prove that the group $\langle a, b | a = [a, a^b] \rangle$ is hyperlinear. Unlike the non residually finite Baumslag-Solitar groups, this group is not residually solvable.

1. Introduction

Let Γ denote the one-relator group $\langle a, b | a^{-1}[a, a^b] \rangle$, where $a^b = bab^{-1}$ and $[a, a^b] = a^{-1}(a^b)^{-1}aa^b$. This group was introduced by G. Baumslag in [**Baum69**] as an example of a non-cyclic one-relator group with the property that all of its finite index quotients are cyclic. It follows that the group Γ is not residually finite. Also, Γ is not residually solvable, since a lies in every one of the derived subgroups of Γ. A countable discrete group G is *hyperlinear* if it can be embedded as a subgroup of the unitary group $U(\mathcal{R}^\omega)$ of an ultrapower $\mathcal{R}^\omega$ of the hyperfinite type II_1 factor $\mathcal{R}$ (cf. [**Pest08**]). Equivalently, G is hyperlinear if the group von Neumann algebra $L(G)$ is embeddable into $\mathcal{R}^\omega$ (cf. [**Pest08**]). Proposition 4.14 of [**Ueda09**], establishes that every HNN extension of an $\mathcal{R}^\omega$-embeddable type II_1 factor over a hyperfinite von Neumann subalgebra is also $\mathcal{R}^\omega$-embeddable. We use this fact along with a now standard trick of McCool and Schupp for one-relator groups to prove that the group Γ above is hyperlinear. The main interest in this example is that it is an example of a non residually solvable hyperlinear one-relator group, and thus our result sheds a little light on the question of Nate Brown asking whether every one-relator group is hyperlinear. In [**Rad00**], Radulescu proved that the non residually finite Baumslag-Solitar group $\langle a, b | ab^3a^{-1}b^{-2} \rangle$ is hyperlinear. Radulescu's result is shown in [**Pest08**] to follow more simply from the fact that these Baumslag-Solitar groups are residually solvable, and hence sofic.

2. Main Result

THEOREM 1. *The group* $\Gamma = \langle a, b | a^{-1}[a, a^b] \rangle$ *is hyperlinear.*

PROOF. We apply a rewriting process due to McCool and Schupp (cf. [**McCSch73**]). Let $a_0 = a$ and $a_{-1} = bab^{-1}$. Note that the word

$$a^{-1}[a, a^b] = a^{-2}ba^{-1}b^{-1}abab^{-1}$$

1991 *Mathematics Subject Classification.* Primary 46L10; Secondary 20F65.
Key words and phrases. sofic group, hyperlinear group, one-relator group.

when rewritten in terms of a_0 and a_{-1} becomes

$$a_0^{-2}(a_{-1})^{-1}a_0(a_{-1}).$$

The group $H = \langle a_0, a_{-1} | a_0^{-2}(a_{-1})^{-1}a_0(a_{-1})\rangle$ is amenable, essentially by the Tits alternative. Or, we may appeal to Theorem 1.2 of [**CeGrig97**], and note that $a_0^{-2}(a_{-1})^{-1}a_0(a_{-1})$ has exponent sum zero on a_{-1}, and can be obtained from $(a_{-1})a_0(a_{-1})^{-1}a_0^{-2}$ by inverting a_{-1} and cyclically shifting, and hence H is amenable. We then note that the group Γ is isomorphic to the HNN extension

$$H*_{\varphi} = \langle t, H | t^{-1}a_{-1}t = a_0\rangle.$$

Now, consider the group von Neumann algebra $L(H*_{\varphi})$. By Corollary 3.5 of [**Ueda05**], this is isomorphic to a reduced HNN extension of the hyperfinite II_1 factor $\mathcal{R}$ over $L(\mathbb{Z})$. Therefore, by Proposition 4.14 of [**Ueda09**], $L(H*_{\varphi})$ is embeddable into an $\mathcal{R}^{\omega}$, and therefore Γ is hyperlinear. □

REMARK 1. *We wish to thank the referee for pointing out that recently it has been shown that any HNN extension of a sofic group over an amenable subgroup is sofic. Precisely, this is Corollary 3.4 of* [**DykCol10**]. *We may, in the above proof, replace Ueda's result by this one and obtain that* Γ *is, in fact, a sofic group.*

References

[Baum69] G. Baumslag, *A non-cyclic one relator group all of whose finite quotients are cyclic,* J. Austral. Math. Soc., Volume 10: (1969), 497-498.

[CeGrig97] T.G. Ceccerini-Silberstein and R. I. Grigorchuk, *Amenability and growth in one-relator groups,* L'Enseignement Mathématique 43 (1997) 337-354.

[DykCol10] K. Dykema and B. Collins, *Free products of sofic groups with amalgamation over amenable groups,* arXiv:math/1003.1675v1, 2010.

[McCSch73] J. McCool and P. Schupp, *On one relator groups and HNN extensions,* J. of the Austral. Math. Soc. Volume 16: (1973), 249-256.

[Pest08] V. Pestov, *Hyperlinear and sofic groups: a brief guide,* Bull. Symbolic Logic Volume 14, Issue 4: (2008), 449-480.

[Rad00] F. Radulescu, *The von Neumann algebra of the non-residually finite Baumslag group* $\langle a, b | ab^3a^{-1}b^{-2}\rangle$ *embeds into* R^{ω}, arXiv:math/0004172v3, 2000.

[Ueda05] Y. Ueda, *HNN extensions of von Neumann algebras,* Journal of Functional Analysis, Volume 225, Issue 2: (2005), 383–426.

[Ueda09] Y. Ueda, *Remarks on HNN extensions in operator algebras,* Illinois J. Math. Volume 52, Issue 3: (2009), 705-725.

DEPARTMENT OF MATHEMATICS, SIENA COLLEGE, LOUDONVILLE NY, 12211.
E-mail address: jbannon@siena.edu
URL: http://www.sos.siena.edu/~jbannon/index.htm